**NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE**
**(NAAC Accredited)**
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

**DEPARTMENT OF MECHATRONICS ENGINEERING**

# COURSE MATERIAL

# MR 306 MECHANICS OF SOLIDS

| VISION OF THE INSTITUTION |
|---|
| To mould true citizens who are millennium leaders and catalysts of change through excellence in education. |
| **MISSION OF THE INSTITUTION** |
| **NCERC** is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.<br><br>We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged. |

## ABOUT DEPARTMENT

- Established in: 2013
- Course offered: B.Tech Mechatronics Engineering
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

## DEPARTMENT VISION

To develop professionally ethical and socially responsible Mechatronics engineers to serve the humanity through quality professional education.

## DEPARTMENT MISSION

1) The department is committed to impart the right blend of knowledge and quality education to create professionally ethical and socially responsible graduates.

2) The department is committed to impart the awareness to meet the current challenges in technology.

3) Establish state-of-the-art laboratories to promote practical knowledge of mechatronics to meet the needs of the society

## PROGRAMME EDUCATIONAL OBJECTIVES

I. Graduates shall have the ability to work in multidisciplinary environment with good professional and commitment.

II. Graduates shall have the ability to solve the complex engineering problems by applying electrical, mechanical, electronics and computer knowledge and engage in lifelong learning in their profession.

III. Graduates shall have the ability to lead and contribute in a team with entrepreneur skills, professional, social and ethical responsibilities.

IV. Graduates shall have ability to acquire scientific and engineering fundamentals necessary for higher studies and research.

**PROGRAM OUTCOME (PO'S)**

**Engineering Graduates will be able to:**

**PO 1. Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

**PO 2. Problem analysis:** Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

**PO 3. Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

**PO 4. Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

**PO 5. Modern tool usage:** Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

**PO 6. The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

**PO 7. Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

**PO 8. Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

**PO 9. Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

**PO 10. Communication:** Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

**PO 11. Project management and finance:** Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO 12. Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

**PROGRAM SPECIFIC OUTCOME(PSO'S)**

**PSO 1:** Design and develop Mechatronics systems to solve the complex engineering problem by integrating electronics, mechanical and control systems.

**PSO 2:** Apply the engineering knowledge to conduct investigations of complex engineering problem related to instrumentation, control, automation, robotics and provide solutions.

## COURSE OUTCOME

**After the completion of the course the student will be able to**

| | |
|---|---|
| CO 1 | Acquire knowledge about the basic concepts of stress and strain in solids |
| CO 2 | Identify and apply the methodologies to analyze stresses and strains at a point |
| CO 3 | Understand the concepts of torsion in elastic circular bars |
| CO 4 | Interpret about the concepts of stresses in beams |
| CO 5 | Identify the concepts of shear force and bending moment in beams |
| CO 6 | Understand about stresses in springs and columns with different conditions |

## CO VS PO'S AND PSO'S MAPPING

| CO | PO1 | PO2 | PO 3 | PO 4 | PO5 | PO6 | PO7 | PO8 | PO9 | PO10 | PO11 | PO12 | PS0 1 | PSO 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CO 1 | 3 | 3 | - | - | - | - | - | - | - | - | - | 2 | 2 | 2 |
| CO 2 | 3 | 3 | - | - | - | - | - | - | - | - | - | 2 | 2 | 2 |
| CO 3 | 3 | 3 | 2 | 3 | - | - | - | - | - | - | - | 2 | 2 | 2 |
| CO 4 | 3 | 3 | 2 | 2 | - | - | - | - | - | - | - | 2 | 2 | 2 |
| CO 5 | 3 | 3 | - | 3 | - | - | - | - | - | - | - | 2 | 2 | 2 |
| CO 6 | 3 | 3 | 2 | 2 | - | - | - | - | - | - | - | 2 | 2 | 2 |

**Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1**

## SYLLABUS

| Course Plan | | | |
|---|---|---|---|
| **Module** | **Contents** | **Hours** | **Sem. Exam Marks** |
| I | Simple Stress and Strain: Introduction to analysis of deformable bodies – internal forces – method of sections – assumptions and limitations. Simple stresses – stresses due to normal- shear and bearing loads – strength design of simple members. Definition of linear and shear strains- Material behavior-stress-strain diagrams. | 7 | 15% |
| II | Hooke's law for linearly elastic isotropic material under axial and shear deformation – deformation in axially loaded bars– statically indeterminate problems – principle of superposition. Elastic strain energy for uniaxial stress. Definition of stress and strain at a point (introduction to stress and strain tensors and its components only) – Poisson's ratio – biaxial deformations – Bulk modulus - Relations between elastic constants. | 7 | 15% |
| III | Torsion: Torsion theory of elastic circular bars – assumptions and limitations – torsional rigidity – economic cross-sections – statically indeterminate problems – shaft design for torsional load. | 7 | 15% |
| IV | Stresses in beams: Pure bending – flexure formula for beams – assumptions and limitations – section modulus - flexural rigidity - economic sections – beam of uniform strength. Shearing stress formula for beams – assumptions and limitations. | 7 | 15% |
| V | Axial force- shear force and bending moment: Diagrammatic conventions for supports and loading - axial force- shear force and bending moment in a beam – differential relations between load- shear force and bending moment - shear force and bending moment diagrams by direct and summation approach – elastic curve – point of inflection. | 7 | 20% |
| VI | Types of springs- stiffness stresses and deflection in helical spring and leaf spring. Columns – Buckling and stiffness due to axial loads – Euler- Rankin and Empirical formulae for columns with different conditions. | 7 | 20% |

# QUESTION BANK

| MODULE I | | | | |
|---|---|---|---|---|
| **Q:NO:** | **QUESTIONS** | **CO** | **KL** | **PAGE NO:** |
| 1 | Explain different types of stresses | CO1 | K1 | 11 |
| 2 | Write in detail about longitudinal stress and strain | CO1 | K2 | 15 |
| 3 | Problems related to stress | CO1 | K5 | 16 |
| 4 | Derive a formula for strain in bars of varying c/s | CO1 | K6 | 20 |
| 5 | Write a short note on Principle of Superposition | CO1 | K1 | 24 |
| 6 | Problems related to superposition theorem | CO1 | K5 | 25 |
| 7 | Problems related to superposition theorem,stress,strain | CO1 | K6 | 31 |
| 8 | Discuss about method of sections | CO1 | K2 | 34 |
| 9 | Write a short note on Hooke's Law | CO1 | K1 | 40 |
| 10 | Discuss about stress-strain curve in ductile & brittle materials | CO1 | K2 | 41 |
| **MODULE II** | | | | |
| 1 | Discuss about stress and strain tensor | CO2 | K2 | 47 |
| 2 | Write a short note on resilience, proof resilience | CO2 | K1 | 49 |
| 3 | What is strain energy | CO2 | K1 | 49 |
| 4 | Derive an equation for strain energy when load is applied gradually | CO2 | K6 | 50 |
| 5 | Derive an equation for strain energy when load is applied suddenly | CO2 | K6 | 51 |
| 6 | Problems on strain energy | CO2 | K5 | 52 |
| 7 | Write down the relation between elastic constants | CO2 | K1 | 56 |
| 8 | Problems on elastic constants | CO2 | K5 | 57 |
| 9 | Problems on strain energy, elastic constants | CO2 | K5 | 59 |
| 10 | Write a short note on statically indeterminate structures | CO2 | K1 | 60 |

| | | | | |
|---|---|---|---|---|
| 2 | Types of loads | CO5 | K1 | 103 |
| 3 | Discuss about sign conventions for SFD & BMD | CO5 | K2 | 104 |
| 4 | Discuss about SFD & BMD in cantilever beam | CO5 | K2 | 106 |
| 5 | Discuss about SFD & BMD in cantilever beam carrying UDL | CO5 | K2 | 109 |
| 6 | Problems related to cantilever beam | CO5 | K5 | 110 |
| 7 | Discuss about SFD & BMD in simply supported beam carrying UDL | CO5 | K2 | 113 |
| 8 | Problems for finding maxing bending moment | CO5 | K5 | 119 |
| 9 | Problems related to point of contra flexure | CO5 | K5 | 121 |
| 10 | Problems related to SFD and BMD | CO5 | K5 | 118 |
| **MODULE VI** | | | | |
| 1 | What is a laminated leaf spring | CO6 | K1 | 123 |
| 2 | Expression for central deflection | CO6 | K6 | 124 |
| 3 | Problem related to central deflection | CO6 | K5 | 125 |
| 4 | Discuss about helical spring | CO6 | K1 | 126 |
| 5 | Expression for deflection I a helical spring | CO6 | K6 | 127 |
| 6 | Problems related to helical spring | CO6 | K5 | 128 |
| 7 | How failure of a column takes place | CO6 | K2 | 129 |
| 8 | Discuss about assumptions in column theory | CO6 | K1 | 130 |
| 9 | Expression for crippling load when both the ends of the column are hinged | CO6 | K6 | 131 |
| 10 | What is meant by equivalent length of a column | CO6 | K1 | 132 |
| 11 | Derive Rankine’s formula | CO6 | K6 | 137 |

## APPENDIX 1

## CONTENT BEYOND THE SYLLABUS

# MODULE 1

## STRESS & STRAIN

### 1.2. STRESS

The force of resistance per unit area, offered by a body against deformation is known as stress. The external force acting on the body is called the *load or force*. The load is applied on the body while the stress is induced in the material of the body. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are equal.

Mathematically stress is written as, $\sigma = \frac{P}{A}$

where $\sigma$ = Stress (also called intensity of stress),

$P$ = External force or load, and

$A$ = Cross-sectional area.

### 1.3. STRAIN

When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to the original dimension is known as strain. Strain is dimensionless.

Strain may be :

1. Tensile strain,
2. Compressive strain,
3. Volumetric strain, and
4. Shear strain.

If there is some increase in length of a body due to external force, then the ratio of increase of length to the original length of the body is known as *tensile strain*. But if there is some decrease in length of the body, then the ratio of decrease of the length of the body to the original length is known as *compressive strain*. The ratio of change of volume of the body to the original volume is known as *volumetric strain*. The strain produced by shear stress is known as shear strain.

### 1.4. TYPES OF STRESSES

The stress may be normal stress or a shear stress.

Normal stress is the stress which acts in a direction perpendicular to the area. It is represented by $\sigma$ (sigma). The normal stress is further divided into tensile stress and compressive stress.

**1.4.1. Tensile Stress.** The stress induced in a body, when subjected to two equal and opposite pulls as shown in Fig. 1.1 ($a$) as a result of which there is an increase in length, is known as tensile stress. The ratio of increase in length to the original length is known as *tensile strain*. The tensile stress acts normal to the area and it pulls on the area.

Let $P$ = Pull (or force) acting on the body,
$A$ = Cross-sectional area of the body,
$L$ = Original length of the body,
$dL$ = Increase in length due to pull $P$ acting on the body,
$\sigma$ = Stress induced in the body, and
$e$ = Strain (*i.e.*, tensile strain).

Fig. 1.1 (*a*) shows a bar subjected to a tensile force $P$ at its ends. Consider a section *x-x*, which divides the bar into two parts. The part left to the section *x-x*, will be in equilibrium if $P$ = Resisting force ($R$). This is shown in Fig. 1.1 (*b*). Similarly the part right to the section *x-x*, will be in equilibrium if $P$ = Resisting force as shown in Fig. 1.1 (*c*). This resisting force per unit area is known as stress or intensity of stress.

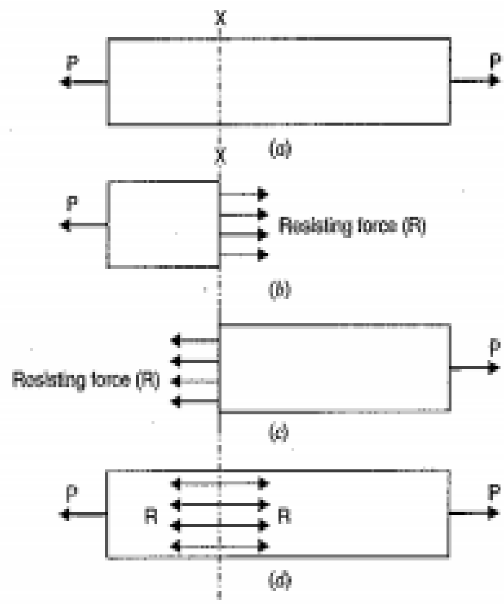


Fig. 1.1

$$\therefore \quad \text{Tensile stress} = \sigma = \frac{\text{Resisting force }(R)}{\text{Cross-sectional area}} = \frac{\text{Tensile load }(P)}{A} \qquad (\because \; P = R)$$

or

$$\sigma = \frac{P}{A} \qquad ...(1.1)$$

And tensile strain is given by,

$$e = \frac{\text{Increase in length}}{\text{Original length}} = \frac{dL}{L}. \qquad ...(1.2)$$

**1.4.2. Compressive Stress.** The stress induced in a body, when subjected to two equal and opposite pushes as shown in Fig. 1.2 (*a*) as a result of which there is a decrease in length of the body, is known as compressive stress. And the ratio of decrease in length to the original length is known as *compressive strain*. The compressive stress acts normal to the area and it pushes on the area.

Let an axial push $P$ is acting on a body is cross-sectional area $A$. Due to external push $P$, let the original length $L$ of the body decreases by $dL$.

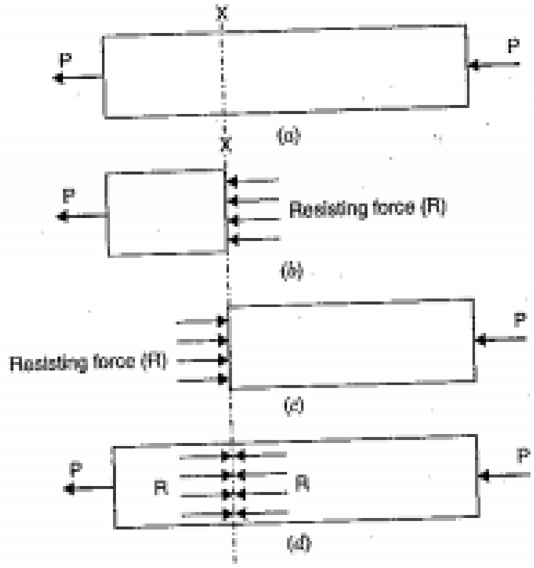



Fig. 1.2

Then compressive stress is given by,

$$\sigma = \frac{\text{Resisting Force } (R)}{\text{Area } (A)} = \frac{\text{Push } (P)}{\text{Area } (A)} = \frac{P}{A}.$$

And compressive strain is given by,

$$e = \frac{\text{Decrease in length}}{\text{Original length}} = \frac{dL}{L}.$$

**1.4.3. Shear Stress.** The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as shown in Fig. 1.3 as a result of which the body tends to shear off across the section, is known as shear stress. The corresponding strain is known as *shear strain*. The shear stress is the stress which acts tangential to the area. It is represented by $\tau$.

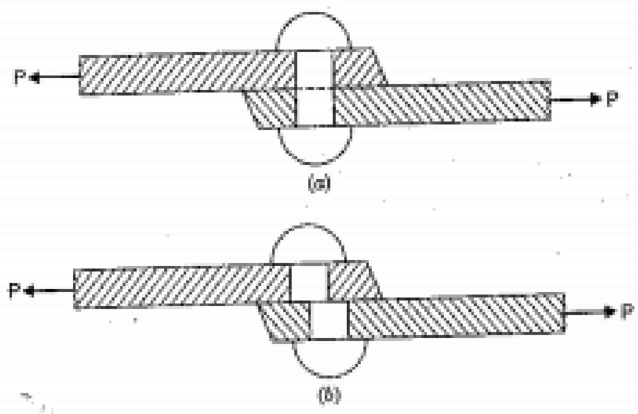



Fig. 1.3

## 1.6. HOOKE'S LAW AND ELASTIC MODULII

Hooke's Law states that when a material is loaded within elastic limit, the stress is proportional to the strain produced by the stress. This means the ratio of the stress to the corresponding strain is a constant within the elastic limit. This constant is known as Modulus of Elasticity or Modulus of Rigidity or Elastic Moduli.

## 1.7. MODULUS OF ELASTICITY (OR YOUNG'S MODULUS)

The ratio of tensile stress or compressive stress to the corresponding strain is a constant. This ratio is known as Young's Modulus or Modulus of Elasticity and is denoted by $E$.

$$\therefore \quad E = \frac{\text{Tensile stress}}{\text{Tensile strain}} \quad \text{or} \quad \frac{\text{Compressive stress}}{\text{Compressive strain}}$$

or

$$E = \frac{\sigma}{e} \qquad \ldots(1.5)$$

**1.7.1. Modulus of Rigidity or Shear Modulus.** The ratio of shear stress to the corresponding shear strain within the elastic limit, is known as Modulus of Rigidity or Shear Modulus. This is denoted by $C$ or $G$ or $N$.

$$\therefore \quad C \text{ (or } G \text{ or } N) = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{\tau}{\phi} \qquad \ldots(1.6)$$

Let us define factor of safety also.

## 1.8. FACTOR OF SAFETY

It is defined as the ratio of ultimate tensile stress to the working (or permissible) stress. Mathematically it is written as

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Permissible stress}} \qquad \ldots(1.7)$$

## 1.9. CONSTITUTIVE RELATIONSHIP BETWEEN STRESS AND STRAIN

**1.9.1. For One-Dimensional Stress System.** The relationship between stress and strain for a unidirectional stress (*i.e.*, for normal stress in one direction only) is given by **Hooke's law,** which states that when a material is loaded within its elastic limit, the normal stress developed is proportional to the strain produced. This means that the ratio of the normal

**1. Longitudinal strain.** When a body is subjected to an axial tensile load, there is an increase in the length of the body. But at the same time there is a decrease in other dimensions of the body at right angles to the line of action of the applied load. Thus the body is having axial deformation and also deformation at right angles to the line of action of the applied load (*i.e.*, lateral deformation).

The ratio of axial deformation to the original length of the body is known as longitudinal (or linear) strain. The longitudinal strain is also defined as the deformation of the body per unit length in the direction of the applied load.

Let $L$ = Length of the body,

$P$ = Tensile force acting on the body,

$\delta L$ = Increase in the length of the body in the direction of $P$.

Then, longitudinal strain $= \dfrac{\delta L}{L}$.

**2. Lateral strain.** The strain at right angles to the direction of applied load is known as lateral strain. Let a rectangular bar of length $L$, breadth $b$ and depth $d$ is subjected to an axial tensile load $P$ as shown in Fig. 1.5. The length of the bar will increase while the breadth and depth will decrease.

Let $\delta L$ = Increase in length,

$\delta b$ = Decrease in breadth, and

$\delta d$ = Decrease in depth.

$$\text{Then longitudinal strain} = \frac{\delta L}{L} \quad \text{...[1.7 (B)]}$$

$$\text{and} \quad \text{lateral strain} = \frac{\delta b}{b} \text{ or } \frac{\delta d}{d} \quad \text{...[1.7 (C)]}$$

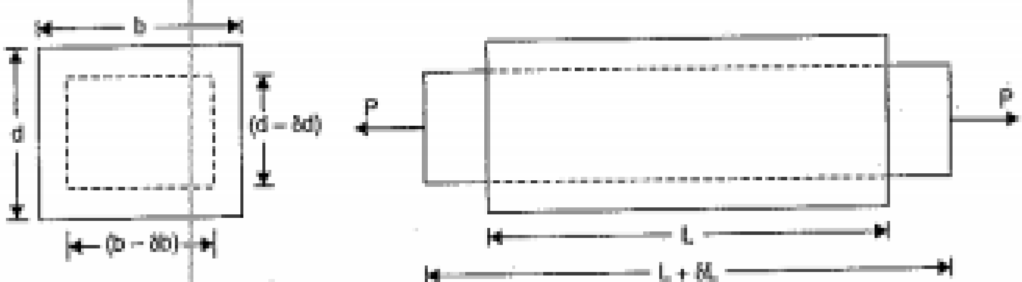



Fig. 1.5

**Note.** (*i*) If longitudinal strain is tensile, the lateral strains will be compressive.

(*ii*) If longitudinal strain is compressive then lateral strains will be tensile.

(*iii*) Hence every longitudinal strain in the direction of load is accompanied by lateral strains of the opposite kind in all directions perpendicular to the load.

**3. Poisson's ratio.** The ratio of lateral strain to the longitudinal strain is a constant for a given material, when the material is stressed within the elastic limit. This ratio is called **Poisson's ratio** and it is generally denoted by μ. Hence mathematically,

$$\text{Poisson's ratio, } \mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} \quad \text{...[1.7 (D)]}$$

or $\quad$ Lateral strain = μ × Longitudinal strain

As lateral strain is opposite in sign to longitudinal strain, hence algebraically, lateral strain is written as

$$\therefore \quad \text{Lateral strain} = -\mu \times \text{Longitudinal strain} \quad \text{...[1.7 (E)]}$$

**Problem 1.2.** *Find the minimum diameter of a steel wire, which is used to raise a load of 4000 N if the stress in the rod is not to exceed 95 MN/m².*

**Sol.** Given : Load, $P = 4000$ N

Stress, $\sigma = 95 \text{ MN/m}^2 = 95 \times 10^6 \text{ N/m}^2$ $\quad (\because \text{ M = Mega} = 10^6)$

$= 95 \text{ N/mm}^2$ $\quad (\because \; 10^6 \text{ N/m}^2 = 1 \text{ N/mm}^2)$

Let $D$ = Diameter of wire in mm

$\therefore$ Area, $A = \frac{\pi}{4} D^2$

Now $\quad \text{stress} = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$

$$95 = \frac{4000}{\frac{\pi}{4} D^2} = \frac{4000 \times 4}{\pi D^2} \quad \text{or} \quad D^2 = \frac{4000 \times 4}{\pi \times 95} = 53.61$$

$\therefore$ $\quad$ **$D$ = 7.32 mm. Ans.**

**Problem 1.3.** *Find the Young's Modulus of a brass rod of diameter 25 mm and of length 250 mm which is subjected to a tensile load of 50 kN when the extension of the rod is equal to 0.3 mm.*

**Sol.** Given : Dia. of rod, $D = 25$ mm

$\therefore$ Area of rod, $A = \frac{\pi}{4} (25)^2 = 490.87 \text{ mm}^2$

Tensile load, $P = 50 \text{ kN} = 50 \times 1000 = 50{,}000$ N

Extension of rod, $dL = 0.3$ mm

Length of rod, $L = 250$ mm

Stress ($\sigma$) is given by equation (1.1), as

$$\sigma = \frac{P}{A} = \frac{50,000}{490.87} = 101.86 \text{ N/mm}^2.$$

Strain ($e$) is given by equation (1.2), as

$$e = \frac{dL}{L} = \frac{0.3}{250} = 0.0012.$$

Using equation (1.5), the Young's Modulus ($E$) is obtained, as

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{101.86 \text{ N/mm}^2}{0.0012} = 84883.33 \text{ N/mm}^2$$

$$= \mathbf{84883.33 \times 10^6\ N/m^2}. \quad \textbf{Ans.} \quad (\because\ 1 \text{ N/mm}^2 = 10^6 \text{ N/m}^2)$$

$$= 84.883 \times 10^9 \text{ N/m}^2 = \mathbf{84.883\ GN/m^2}. \quad \textbf{Ans.} \quad (\because\ 10^9 = G)$$

**Problem 1.4.** *A tensile test was conducted on a mild steel bar. The following data was obtained from the test :*

| | |
|---|---|
| *(i) Diameter of the steel bar* | *= 3 cm* |
| *(ii) Gauge length of the bar* | *= 20 cm* |
| *(iii) Load at elastic limit* | *= 250 kN* |
| *(iv) Extension at a load of 150 kN* | *= 0.21 mm* |
| *(v) Maximum load* | *= 380 kN* |
| *(vi) Total extension* | *= 60 mm* |
| *(vii) Diameter of the rod at the failure* | *= 2.25 cm.* |

*Determine :(a) the Young's modulus, (b) the stress at elastic limit, (c) the percentage elongation, and (d) the percentage decrease in area.*

**Sol.** Area of the rod, $A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (3)^2 \text{ cm}^2$

$$= 7.0685 \text{ cm}^2 = 7.0685 \times 10^{-4} \text{ m}^2. \quad \left[\because\ \text{cm}^2 = \left(\frac{1}{100} \text{ m}\right)^2\right]$$

(a) To find Young's modulus, first calculate the value of stress and strain within elastic limit. The load at elastic limit is given but the extension corresponding to the load at elastic limit is not given. But a load of 150 kN (which is within elastic limit) and corresponding extension of 0.21 mm are given. Hence these values are used for stress and strain within elastic limit

$$\therefore \quad \text{Stress} = \frac{\text{Load}}{\text{Area}} = \frac{150 \times 1000}{7.0685 \times 10^{-4}} \text{ N/m}^2 \quad (\because\ 1 \text{ kN} = 1000 \text{ N})$$

$$= 21220.9 \times 10^4 \text{ N/m}^2$$

and

$$\text{Strain} = \frac{\text{Increase in length (or Extension)}}{\text{Original length (or Gauge length)}}$$

$$= \frac{0.21 \text{ mm}}{20 \times 10 \text{ mm}} = 0.00105$$

$\therefore$ Young's Modulus,

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{21220.9 \times 10^4}{0.00105} = 20209523 \times 10^4 \text{ N/m}^2$$

$= 202.095 \times 10^9$ N/m$^2$ $\quad (\because \; 10^9 = \text{Giga} = G)$

$= \mathbf{202.095 \; GN/m^2}$. **Ans.**

(b) The stress at the elastic limit is given by,

$$\text{Stress} = \frac{\text{Load at elastic limit}}{\text{Area}} = \frac{250 \times 1000}{7.0685 \times 10^{-4}}$$

$= 35368 \times 10^4$ N/m$^2$

$= 353.68 \times 10^6$ N/m$^2$ $\quad (\because \; 10^6 = \text{Mega} = M)$

$= \mathbf{353.68 \; MN/m^2}$. **Ans.**

(c) The percentage elongation is obtained as,

Percentage elongation

$$= \frac{\text{Total increase in length}}{\text{Original length (or Gauge length)}} \times 100$$

$$= \frac{60 \text{ mm}}{20 \times 10 \text{ mm}} \times 100 = \mathbf{30\%}. \text{ Ans.}$$

(d) The percentage decrease in area is obtained as,

Percentage decrease in area

$$= \frac{(\text{Original area} - \text{Area at the failure})}{\text{Original area}} \times 100$$

$$= \frac{\left(\frac{\pi}{4} \times 3^2 - \frac{\pi}{4} \times 2.25^2\right)}{\frac{\pi}{4} \times 3^2} \times 100$$

$$= \left(\frac{3^2 - 2.25^2}{3^2}\right) \times 100 = \frac{(9 - 5.0625)}{9} \times 100 = \mathbf{43.75\%}. \text{ Ans.}$$

**Problem 1.5.** *The safe stress, for a hollow steel column which carries an axial load of $2.1 \times 10^3$ kN is 125 MN/m$^2$. If the external diameter of the column is 30 cm, determine the internal diameter.*

**Sol.** Given :

Safe stress*, $\quad \sigma = 125$ MN/m$^2$ $= 125 \times 10^6$ N/m$^2$

Axial load, $\quad P = 2.1 \times 10^3$ kN $= 2.1 \times 10^6$ N

External diameter, $\quad D = 30$ cm $= 0.30$ m

Let $\quad d$ = Internal diameter

$\therefore$ Area of cross-section of the column,

$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(.30^2 - d^2) \text{ m}^2$$

Using equation (1.1), $\quad \sigma = \dfrac{P}{A}$

or $$125 \times 10^6 = \frac{2.1 \times 10^6}{\frac{\pi}{4}(.30^2 - d^2)} \quad \text{or} \quad (.30^2 - d^2) = \frac{4 \times 2.1 \times 10^6}{\pi \times 125 \times 10^6}$$

or $$0.09 - d^2 = 213.9 \quad \text{or} \quad 0.09 - 0.02139 = d^2$$

$$\therefore \quad d = \sqrt{0.09 - 0.02139} = 0.2619 \text{ m} = \mathbf{26.19 \text{ cm}}. \quad \textbf{Ans.}$$

**Problem 1.6.** *The ultimate stress, for a hollow steel column which carries an axial load of 1.9 MN is 480 N/mm². If the external diameter of the column is 200 mm, determine the internal diameter. Take the factor of safety as 4.*

**Sol.** Given :

Ultimate stress, $= 480 \text{ N/mm}^2$

Axial load, $P = 1.9 \text{ MN} = 1.9 \times 10^6 \text{ N}$ $(\because \; M = 10^6)$

$= 1900000 \text{ N}$

External dia., $D = 200 \text{ mm}$

Factor of safety $= 4$

Let $d$ = Internal diameter in mm

$\therefore$ Area of cross-section of the column,

$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(200^2 - d^2) \text{ mm}^2$$

Using equation (1.7), we get

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Working stress or Permissible stress}}$$

$$\therefore \quad 4 = \frac{480}{\text{Working stress}}$$

or $$\text{Working stress} = \frac{480}{4} = 120 \text{ N/mm}^2$$

$$\therefore \quad \sigma = 120 \text{ N/mm}^2$$

Now using equation (1.1), we get

$$\sigma = \frac{P}{A} \quad \text{or} \quad 120 = \frac{1900000}{\frac{\pi}{4}(200^2 - d^2)} = \frac{1900000 \times 4}{\pi(40000 - d^2)}$$

or $$40000 - d^2 = \frac{1900000 \times 4}{\pi \times 120} = 20159.6$$

or $$d^2 = 40000 - 20159.6 = 19840.4$$

$$\therefore \quad d = \mathbf{140.85 \text{ mm}}. \quad \textbf{Ans.}$$

**Problem 1.7.** *A stepped bar shown in Fig. 1.6 is subjected to an axially applied compressive load of 35 kN. Find the maximum and minimum stresses produced.*

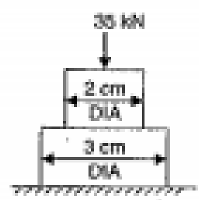


Fig. 1.6

**Sol.** Given :

Axial load, $P = 35 \text{ kN} = 35 \times 10^3 \text{ N}$

Dia. of upper part, $D_1 = 2 \text{ cm} = 20 \text{ mm}$

$\therefore$ Area of upper part, $A_1 = \frac{\pi}{4}(20^2) = 100\,\pi\ mm^2$

Area of lower part, $A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4}(30^2) = 225\,\pi\ mm^2$

The stress is equal to load divided by area. Hence stress will be maximum where area is minimum. Hence stress will be maximum in upper part and minimum in lower part.

$\therefore$ Maximum stress $= \frac{\text{Load}}{A_1} = \frac{35 \times 10^3}{100 \times \pi} =$ **111.408 N/mm². Ans.**

Minimum stress $= \frac{\text{Load}}{A_2} = \frac{35 \times 10^3}{225 \times \pi} =$ **49.5146 N/mm². Ans.**

## 1.10. ANALYSIS OF BARS OF VARYING SECTIONS

A bar of different lengths and of different diameters (and hence of different cross-sectional areas) is shown in Fig. 1.6 (*a*). Let this bar is subjected to an axial load *P*.

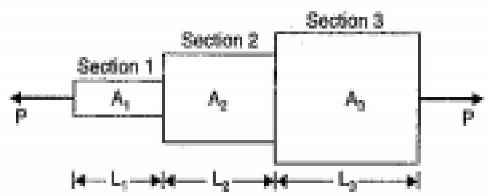



Fig. 1.6 (*a*)

Though each section is subjected to the same axial load *P*, yet the stresses, strains and change in lengths will be different. The total change in length will be obtained by adding the changes in length of individual section.

Let $P$ = Axial load acting on the bar,

$L_1$ = Length of section 1,

$A_1$ = Cross-sectional area of section 1,

$L_2, A_2$ = Length and cross-sectional area of section 2,

$L_3, A_3$ = Length and cross-sectional area of section 3, and

$E$ = Young's modulus for the bar.

Then stress for the section 1,

$$\sigma_1 = \frac{\text{Load}}{\text{Area of section 1}} = \frac{P}{A_1}.$$

Similarly stresses for the section 2 and section 3 are given as,

$$\sigma_2 = \frac{P}{A_2} \quad \text{and} \quad \sigma_3 = \frac{P}{A_3}$$

Using equation (1.5), the strains in different sections are obtained.

$\therefore$ Strain of section 1, $e_1 = \frac{\sigma_1}{E} = \frac{P}{A_1 E}$ $\qquad \left(\because \sigma_1 = \frac{P}{A_1}\right)$

Similarly the strains of section 2 and of section 3 are,

$$e_2 = \frac{\sigma_2}{E} = \frac{P}{A_2 E} \quad \text{and} \quad e_3 = \frac{\sigma_3}{E} = \frac{P}{A_3 E}.$$

But strain in section 1 $= \dfrac{\text{Change in length of section 1}}{\text{Length of section 1}}$

or
$$e_1 = \frac{dL_1}{L_1}$$

where $dL_1$ = change in length of section 1.

∴ Change in length of section 1, $dL_1 = e_1 L_1$

$$= \frac{PL_1}{A_1 E} \qquad \left(\because e_1 = \frac{P}{A_1 E}\right)$$

Similarly changes in length of section 2 and of section 3 are obtained as :

Change in length of section 2, $dL_2 = e_2 L_2$

$$= \frac{PL_2}{A_2 E} \qquad \left(\because e_2 = \frac{P}{A_2 E}\right)$$

and change in length of section 3, $dL_3 = e_3 L_3$

$$= \frac{PL_3}{A_3 E} \qquad \left(\because e_3 = \frac{P}{A_3 E}\right)$$

∴ Total change in the length of the bar,

$$dL = dL_1 + dL_2 + dL_3 = \frac{PL_1}{A_1 E} + \frac{PL_2}{A_2 E} + \frac{PL_3}{A_3 E}$$

$$= \frac{P}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right] \qquad ...(1.8)$$

Equation (1.8) is used when the Young's modulus of different sections is same. If the Young's modulus of different sections is different, then total change in length of the bar is given by,

$$dL = P\left[\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} + \frac{L_3}{E_3 A_3}\right] \qquad ...(1.9)$$

**Problem 1.8.** *An axial pull of 35000 N is acting on a bar consisting of three lengths as shown in Fig. 1.6 (b). If the Young's modulus = $2.1 \times 10^5$ N/mm², determine :*

*(i) stresses in each section and*

*(ii) total extension of the bar.*

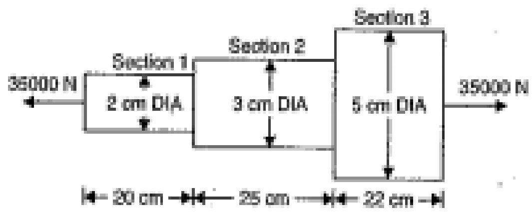



Fig. 1.6 (b)

**Sol.** Given :

Axial pull, $P = 35000$ N

Length of section 1, $L_1 = 20$ cm $= 200$ mm

Dia. of section 1, $D_1 = 2$ cm $= 20$ mm

$\therefore$ Area of section 1, $A_1 = \frac{\pi}{4}(20^2) = 100\,\pi$ mm$^2$

Length of section 2, $L_2 = 25$ cm $= 250$ mm

Dia. of section 2, $D_2 = 3$ cm $= 30$ mm

$\therefore$ Area of section 2, $A_2 = \frac{\pi}{4}(30^2) = 225\,\pi$ mm$^2$

Length of section 3, $L_3 = 22$ cm $= 220$ mm

Dia. of section 3, $D_3 = 5$ cm $= 50$ mm

$\therefore$ Area of section 3, $A_3 = \frac{\pi}{4}(50^2) = 625\,\pi$ mm$^2$

Young's modulus, $E = 2.1 \times 10^5$ N/mm$^2$.

*(i) Stresses in each section*

Stress in section 1, $\sigma_1 = \frac{\text{Axial load}}{\text{Area of section 1}}$

$$= \frac{P}{A_1} = \frac{35000}{100\,\pi} = \mathbf{111.408\ N/mm^2.}\quad \textbf{Ans.}$$

Stress in section 2, $\sigma_2 = \frac{P}{A_2} = \frac{35000}{225 \times \pi} = \mathbf{49.5146\ N/mm^2.}$ **Ans.**

Stress in section 3, $\sigma_3 = \frac{P}{A_3} = \frac{35000}{625 \times \pi} = \mathbf{17.825\ N/mm^2.}$ **Ans.**

*(ii) Total extension of the bar*

Using equation (1.8), we get

Total extension $= \frac{P}{E}\left(\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right)$

$$= \frac{35000}{2.1 \times 10^5}\left(\frac{200}{100\,\pi} + \frac{250}{225 \times \pi} + \frac{220}{625 \times \pi}\right)$$

$$= \frac{35000}{2.1 \times 10^5}(6.366 + 3.536 + 1.120) = \mathbf{0.183\ mm.}\quad \textbf{Ans.}$$

**Problem 1.9.** *A member formed by connecting a steel bar to an aluminium bar is shown in Fig. 1.7. Assuming that the bars are prevented from buckling sideways, calculate the magnitude of force P that will cause the total length of the member to decrease 0.25 mm. The values of elastic modulus for steel and aluminium are $2.1 \times 10^5$ N/mm$^2$ and $7 \times 10^4$ N/mm$^2$ respectively.*

**Sol.** Given :

Length of steel bar, $L_1 = 30$ cm $= 300$ mm

Area of steel bar, $A_1 = 5 \times 5 = 25 \text{ cm}^2 = 250 \text{ mm}^2$

Elastic modulus for steel bar,

$$E_1 = 2.1 \times 10^5 \text{ N/mm}^2$$

Length of aluminium bar,

$$L_2 = 38 \text{ cm} = 380 \text{ mm}$$

Area of aluminium bar,

$$A_2 = 10 \times 10 = 100 \text{ cm}^2 = 10000 \text{ mm}^2$$

Elastic modulus for aluminium bar,

$$E_2 = 7 \times 10^4 \text{ N/mm}^2$$

Total decrease in length, $dL = 0.25$ mm

Let $P$ = Required force.

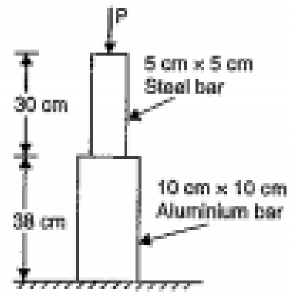



Fig. 1.7

As both the bars are made of different materials, hence total change in the lengths of the bar is given by equation (1.9).

$$\therefore \quad dL = P\left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2}\right)$$

or

$$0.25 = P\left(\frac{300}{2.1 \times 10^5 \times 2500} + \frac{380}{7 \times 10^4 \times 10000}\right)$$

$$= P\,(5.714 \times 10^{-7} + 5.428 \times 10^{-7}) = P \times 11.142 \times 10^{-7}$$

$$\therefore \quad P = \frac{0.25}{11.142 \times 10^{-7}} = \frac{0.25 \times 10^7}{11.142}$$

$= 2.2437 \times 10^5 =$ **224.37 kN. Ans.**

**Problem 1.10.** *The bar shown in Fig. 1.8 is subjected to a tensile load of 160 kN. If the stress in the middle portion is limited to 150 N/mm², determine the diameter of the middle portion. Find also the length of the middle portion if the total elongation of the bar is to be 0.2 mm. Young's modulus is given as equal to 2.1 × 10⁵ N/mm².*

**Sol.** Given :

Tensile load, $P = 160 \text{ kN} = 160 \times 10^3 \text{ N}$

Stress in middle portion, $\sigma_2 = 150 \text{ N/mm}^2$

Total elongation, $dL = 0.2$ mm

Total length of the bar, $L = 40 \text{ cm} = 400 \text{ mm}$

Young's modulus, $E = 2.1 \times 10^5 \text{ N/mm}^2$

Diameter of both end portions, $D_1 = 6 \text{ cm} = 60 \text{ mm}$

$\therefore$ Area of cross-section of both end portions,

$$A_1 = \frac{\pi}{4} \times 60^2 = 900\,\pi \text{ mm}^2.$$

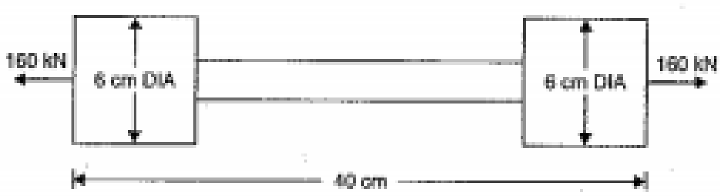



Fig. 1.8

Let $D_2$ = Diameter of the middle portion

$L_2$ = Length of middle portion in mm.

$\therefore$ Length of both end portions of the bar,

$$L_1 = (400 - L_2) \text{ mm}$$

Using equation (1.1), we have

$$\text{Stress} = \frac{\text{Load}}{\text{Area}}.$$

For the middle portion, we have

$$\sigma_2 = \frac{P}{A_2} \qquad \text{where } A_2 = \frac{\pi}{4} D_2^2$$

or
$$150 = \frac{160000}{\frac{\pi}{4} D_2^2}$$

$\therefore$
$$D_2^2 = \frac{4 \times 160000}{\pi \times 150} = 1358 \text{ mm}^2$$

or
$$D_2 = \sqrt{1358} = 36.85 \text{ mm} = \mathbf{3.685 \text{ cm.}} \quad \textbf{Ans.}$$

$\therefore$ Area of cross-section of middle portion,

$$A_2 = \frac{\pi}{4} \times 36.85 = 1066 \text{ mm}^2$$

Now using equation (1.8), we get

Total extension,
$$dL = \frac{P}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2}\right]$$

or
$$0.2 = \frac{160000}{2.1 \times 10^5}\left[\frac{(400 - L_2)}{900\pi} + \frac{L_2}{1066}\right]$$

[$\because$ $L_1 = (400 - L_2)$ and $A_2 = 1066$]

or
$$\frac{0.2 \times 2.1 \times 10^5}{160000} = \frac{(400 - L_2)}{900\,\pi} + \frac{L_2}{1066}$$

or
$$0.2625 = \frac{1066(400 - L_2) + 900\,\pi\, L_2}{900\,\pi \times 1066}$$

or
$$0.2625 \times 900\pi \times 1066 = 1066 \times 400 - 1066\, L_2 + 900\pi \times L_2$$

or
$$791186 = 426400 - 1066\, L_2 + 2827\, L_2$$

or
$$791186 - 426400 = L_2\,(2827 - 1066)$$

or
$$364786 = 1761\, L_2$$

$\therefore$
$$L_2 = \frac{364786}{1761} = 207.14 \text{ mm} = \mathbf{20.714 \text{ cm.}} \quad \textbf{Ans.}$$

**1.10.1. Principle of Superposition.** When a number of loads are acting on a body, the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads.

While using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body, first the free body diagram of individual section is drawn. Then the deformation of the each section is obtained. The total deformation of the body will be then equal to the algebraic sum of deformations of the individual sections.

**Problem 1.11.** *A brass bar, having cross-sectional area of 1000 mm², is subjected to axial forces as shown in Fig. 1.9.*

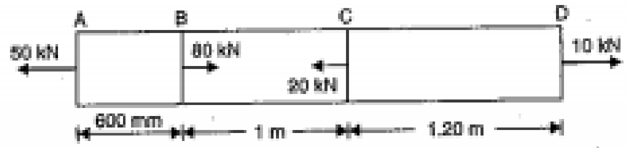



Fig. 1.9

*Find the total elongation of the bar. Take* $E = 1.05 \times 10^5$ *N/mm²*.

**Sol.** Given :

Area, $A = 1000 \text{ mm}^2$

Value of $E = 1.05 \times 10^5 \text{ N/mm}^2$

Let $dL$ = Total elongation of the bar.

The force of 80 kN acting at $B$ is split up into three forces of 50 kN, 20 kN and 10 kN. Then the part $AB$ of the bar will be subjected to a tensile load of 50 kN, part $BC$ is subjected to a compressive load of 20 kN and part $BD$ is subjected to a compressive load of 10 kN as shown in Fig. 1.10.

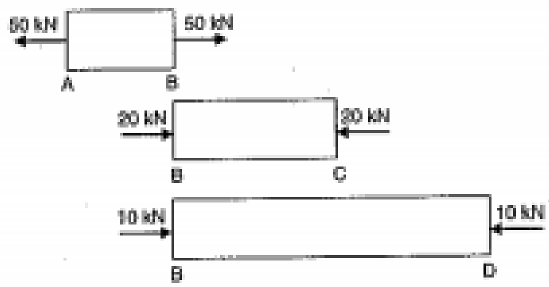



Fig. 1.10

**Part AB.** This part is subjected to a tensile load of 50 kN. Hence there will be increase in length of this part.

∴ Increase in the length of $AB$

$$= \frac{P_1}{AE} \times L_1$$

$$= \frac{50 \times 1000}{1000 \times 1.05 \times 10^5} \times 600 \qquad (\because \; P_1 = 50{,}000 \text{ N}, L_1 = 600 \text{ mm})$$

$$= 0.2857.$$

**Part BC.** This part is subjected to a compressive load of 20 kN or 20,000 N. Hence there will be decrease in length of this part.

∴ Decrease in the length of $BC$

$$= \frac{P_2}{AE} \times L_2 = \frac{20{,}000}{1000 \times 1.05 \times 10^5} \times 1000 \qquad (\because \; L_2 = 1 \text{ m} = 1000 \text{ mm})$$

$$= 0.1904.$$

**Part BD.** This part is subjected to a compressive load of 10 kN or 10,000 N. Hence there will be decrease in length of this part.

∴ Decrease in the length of $BD$

$$= \frac{P_3}{AE} \times L_3 = \frac{10000}{1000 \times 1.05 \times 10^5} \times 2200$$

(∵ $L_3 = 1.2 + 1 = 2.2$ m or 2200 mm)

$= 0.2095.$

∴ Total elongation of bar $= 0.2857 - 0.1904 - 0.2095$

(Taking +ve sign for increase in length and −ve sign for decrease in length)

$= -$ **0.1142 mm. Ans.**

Negative sign shows, that there will be decrease in length of the bar.

**Problem 1.12.** *A member ABCD is subjected to point loads $P_1$, $P_2$, $P_3$ and $P_4$ as shown in Fig. 1.11.*

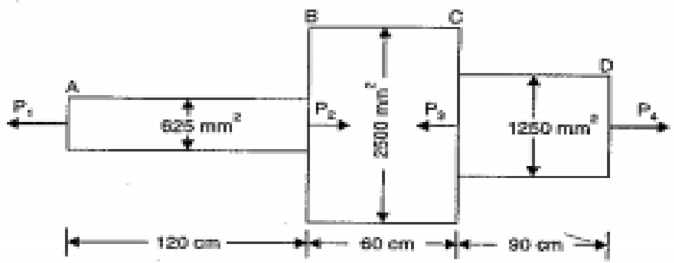



Fig. 1.11

*Calculate the force $P_2$ necessary for equilibrium, if $P_1$ = 45 kN, $P_3$ = 450 kN and $P_4$ = 130 kN. Determine the total elongation of the member, assuming the modulus of elasticity to be $2.1 \times 10^5$ N/mm².*

**Sol.** Given :

| | | |
|---|---|---|
| Part $AB$ : | Area, | $A_1 = 625$ mm² and |
| | Length, | $L_1 = 120$ cm $= 1200$ mm |
| Part $BC$ : | Area, | $A_2 = 2500$ mm² and |
| | Length, | $L_2 = 60$ cm $= 600$ mm |
| Part $CD$ : | Area, | $A_3 = 12.0$ mm² and |
| | Length, | $L_3 = 90$ cm $= 900$ mm |
| Value of | | $E = 2.1 \times 10^5$ N/mm². |

*Value of $P_2$ necessary for equilibrium*

Resolving the forces on the rod along its axis (*i.e.*, equating the forces acting towards right to those acting towards left), we get

$$P_1 + P_3 = P_2 + P_4$$

But $P_1 = 45$ kN,

$P_3 = 450$ kN and $P_4 = 130$ kN

$\therefore \quad 45 + 450 = P_2 + 130 \quad$ or $\quad P_2 = 495 - 130 = 365$ kN

The force of 365 kN acting at $B$ is split into two forces of 45 kN and 320 kN (*i.e.*, 365 – 45 = 320 kN).

The force of 450 kN acting at $C$ is split into two forces of 320 kN and 130 kN (*i.e.*, 450 – 320 = 130 kN) as shown in Fig. 1.12.

From Fig. 1.12, it is clear that part $AB$ is subjected to a tensile load of 45 kN, part $BC$ is subjected to a compressive load of 320 kN and part $CD$ is subjected to a tensile load 130 kN.

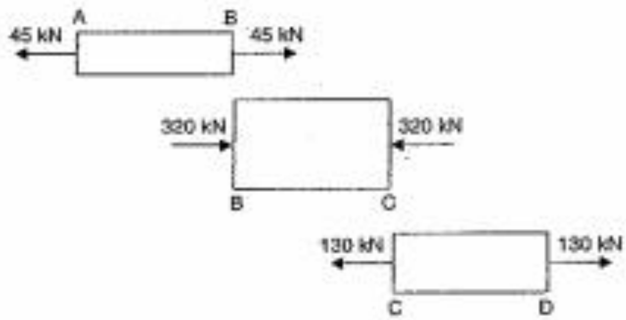



Fig. 1.12

Hence for part $AB$, there will be increase in length ; for part $BC$ there will be decrease in length and for part $CD$ there will be increase in length.

$\therefore$ Increase in length of $AB$

$$= \frac{P}{A_1E} \times L_1 = \frac{45000}{625 \times 2.1 \times 10^5} \times 1200 \quad (\because P = 45 \text{ kN} = 45000 \text{ N})$$

$= 0.4114$ mm

Decrease in length of $BC$

$$= \frac{P}{A_2E} \times L_2 = \frac{320{,}000}{2500 \times 2.1 \times 10^5} \times 600 \quad (\because P = 320 \text{ kN} = 320000)$$

$= 0.3657$ mm

Increase in length of $CD$

$$= \frac{P}{A_3E} \times L_3 = \frac{130{,}000}{1250 \times 2.1 \times 10^5} \times 900 \quad (\because P = 130 \text{ kN} = 130000)$$

$= 0.4457$ mm

Total change in the length of member

$= 0.4114 - 0.3657 + 0.4457$

(Taking +ve sign for increase in length and –ve sign for decrease in length)

**= 0.4914 mm (extension). Ans.**

**Problem 1.13.** *A tensile load of 40 kN is acting on a rod of diameter 40 mm and of length 4 m. A bore of diameter 20 mm is made centrally on the rod. To what length the rod* *should be bored so that the total extension will increase 30% under the same tensile load. Take $E = 2 \times 10^5$ N/mm².*

**Sol.** Given :


Fig. 1.12 (*a*)

Tensile load, $P = 40$ kN $= 40{,}000$ N

Dia. of rod, $D = 40$ mm

∴ Area of rod, $A = \frac{\pi}{4}(40^2) = 400\pi$ mm²

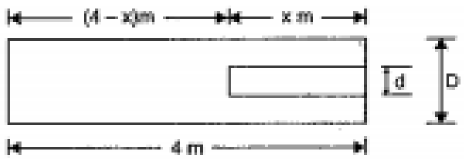



Fig. 1.12 (*b*)

Length of rod, $L = 4$ m $= 4 \times 1000 = 4000$ mm

Dia. of bore, $d = 20$ mm

∴ Area of bore, $a = \frac{\pi}{4} \times 20^2 = 100\,\pi$ mm²

Total extension after bore $= 1.3 \times$ Extension before bore

Value of $E$ $= 2 \times 10^5$ N/mm²

Let the rod be bored to a length of $x$ meter or $x \times 1000$ mm. Then length of unbored portion $= (4 - x)$ m $= (4 - x) \times 1000$ mm. First calculate the extension before the bore is made.

The extension ($\delta L$) is given by,

$$\delta L = \frac{P}{AE} \times L = \frac{40000 \times 4000}{400\pi \times 2 \times 10^5} = \frac{2}{\pi} \text{ mm}$$

Now extension after the bore is made

$$= 1.3 \times \text{Extension before bore}$$

$$= 1.3 \times \frac{2}{\pi} = \frac{2.6}{\pi} \text{ mm} \quad \ldots(i)$$

The extension after the bore is made, is also obtained by finding the extensions of the unbored length and bored length.

For this, find the stresses in the bored and unbored portions.

Stress in unbored portion

$$= \frac{\text{Load}}{\text{Area}} = \frac{P}{A} = \frac{40000}{400\pi} = \frac{100}{\pi} \text{ N/mm}^2$$

∴ Extension of unbored portion

$$= \frac{\text{Stress}}{E} \times \text{Length of unbored portion}$$

$$= \frac{100}{\pi \times 2 \times 10^5} \times (4 - x) \times 1000 = \frac{(4 - x)}{2\pi} \text{ mm}$$

Stress in bored portion

$$= \frac{\text{Load}}{\text{Area}} = \frac{P}{(A - a)} = \frac{40000}{(400\pi - 100\pi)} = \frac{40000}{300\pi}$$

∴ Extension of bored portion

$$= \frac{\text{Stress}}{E} \times \text{Length of bored portion}$$

$$= \frac{40000}{300\pi \times 2 \times 10^5} \times 1000x = \frac{4x}{6\pi} \text{ mm}$$

∴ Total extension after the bore is made

$$= \frac{(4 - x)}{2\pi} + \frac{4x}{6\pi} \qquad ...(ii)$$

Equating the equations (*i*) and (*ii*),

$$\frac{2.6}{\pi} = \frac{4 - x}{2\pi} + \frac{4x}{6\pi}$$

or $2.6 = \frac{4 - x}{2} + \frac{4x}{6}$ or $2.6 \times 6 = 3 \times (4 - x) + 4x$

or $15.6 = 12 - 3x + 4x$ or $15.6 - 12 = x$ or $3.6 = x$

∴ Rod should be bored upto a length of **3.6 m. Ans.**

## 1.13. ANALYSIS OF BARS OF COMPOSITE SECTIONS

A bar, made up of two or more bars of equal lengths but of different materials rigidly fixed with each other and behaving as one unit for extension or compression when subjected to an axial tensile or compressive loads, is called a composite bar. For the composite bar the following two points are important :

1. The extension or compression in each bar is equal. Hence deformation per unit length *i.e.*, strain in each bar is equal.

2. The total external load on the composite bar is equal to the sum of the loads carried by each different material.

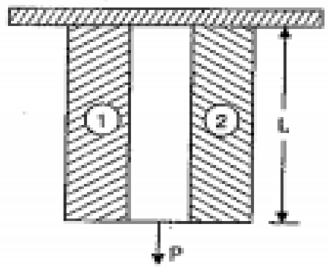


Fig. 1.15

Fig. 1.15 shows a composite bar made up of two different materials.

Let $P$ = Total load on the composite bar,
$L$ = Length of composite bar and also length of bars of different materials,
$A_1$ = Area of cross-section of bar 1,
$A_2$ = Area of cross-section of bar 2,
$E_1$ = Young's Modulus of bar 1,
$E_2$ = Young's Modulus of bar 2,
$P_1$ = Load shared by bar 1,
$P_2$ = Load shared by bar 2,
$\sigma_1$ = Stress induced in bar 1, and
$\sigma_2$ = Stress induced in bar 2.

Now the total load on the composite bar is equal to the sum of the load carried by the two bars.

$$\therefore \quad P = P_1 + P_2 \quad \text{...(i)}$$

The stress in bar 1, $= \dfrac{\text{Load carried by bar 1}}{\text{Area of cross-section of bar 1}}$.

$$\therefore \quad \sigma_1 = \frac{P_1}{A_1} \quad \text{or} \quad P_1 = \sigma_1 A_1 \quad \text{...(ii)}$$

Similarly stress in bar 2, $\sigma_2 = \dfrac{P_2}{A_2}$ or $P_2 = \sigma_2 A_2$ ...(iii)

Substituting the values of $P_1$ and $P_2$ in equation (*i*), we get

$$P = \sigma_1 A_1 + \sigma_2 A_2 \quad \text{...(iv)}$$

Since the ends of the two bars are rigidly connected, each bar will change in length by the same amount. Also the length of each bar is same and hence the ratio of change in length to the original length (*i.e.*, strain) will be same for each bar.

But strain in bar 1, $= \dfrac{\text{Stress in bar 1}}{\text{Young's modulus of bar 1}} = \dfrac{\sigma_1}{E_1}$.

Similarly strain in bar 2, $= \dfrac{\sigma_2}{E_2}$.

But strain in bar 1 = Strain in bar 2

$$\therefore \qquad = \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \qquad \ldots(v)$$

From equations (*iv*) and (*v*), the stresses $\sigma_1$ and $\sigma_2$ can be determined. By substituting the values of $\sigma_1$ and $\sigma_2$ in equations (*ii*) and (*iii*), the load carried by different materials may be computed.

**Modular Ratio.** The ratio of $\frac{E_1}{E_2}$ is called the modular ratio of the first material to the second.

**Problem 1.19.** *A steel rod of 3 cm diameter is enclosed centrally in a hollow copper tube of external diameter 5 cm and internal diameter of 4 cm. The composite bar is then subjected to an axial pull of 45000 N. If the length of each bar is equal to 15 cm, determine :*

*(i) The stresses in the rod and tube, and*

*(ii) Load carried by each bar.*

*Take E for steel = 2.1 × 10⁵ N/mm² and for copper = 1.1 × 10⁵ N/mm².*

**Sol.** Given :

Dia. of steel rod = 3 cm = 30 mm

∴ Area of steel rod,

$$A_s = \frac{\pi}{4}(30)^2 = 706.86 \text{ mm}^2$$

External dia. of copper tube

= 5 cm = 50 mm

Internal dia. of copper tube

= 4 cm = 40 mm

∴ Area of copper tube,

$$A_c = \frac{\pi}{4}[50^2 - 40^2] \text{ mm}^2 = 706.86 \text{ mm}^2$$

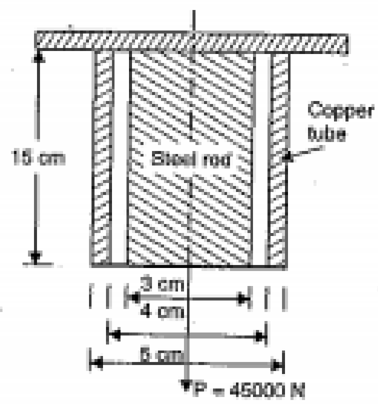



Fig. 1.16

Axial pull on composite bar, $P = 45000$ N

Length of each bar, $L = 15$ cm

Young's modulus for steel, $E_s = 2.1 \times 10^5$ N/mm²

Young's modulus for copper, $E_c = 1.1 \times 10^5$ N/mm²

*(i) The stress in the rod and tube*

Let $\sigma_s$ = Stress in steel,

$P_s$ = Load carried by steel rod,

$\sigma_c$ = Stress in copper, and

$P_c$ = Load carried by copper tube.

Now strain in steel = Strain in copper

$$\text{or} \qquad \frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c} \qquad \left(\because \frac{\sigma}{E} = \text{strain}\right)$$

$$\therefore \qquad \sigma_s = \frac{E_s}{E_c} \times \sigma_c = \frac{2.1 \times 10^5}{11 \times 10^5} \times \sigma_c = 1.909\,\sigma_c \qquad \ldots(i)$$

Now $\quad \text{stress} = \frac{\text{Load}}{\text{Area}}$, $\quad \therefore$ Load = Stress × Area

Load on steel + Load on copper = Total load

$\sigma_s \times A_s + \sigma_c \times A_c = P$ $\quad$ ($\because$ Total load = $P$)

or $\quad 1.909\, \sigma_c \times 706.86 + \sigma_c \times 706.86 = 45000$

or $\quad \sigma_c (1.909 \times 706.86 + 706.86) = 45000$

or $\quad 2056.25\, \sigma_c = 45000$

$\therefore \quad \sigma_c = \frac{45000}{2056.25} = $ **21.88 N/mm². Ans.**

Substituting the value of $\sigma_c$ in equation (*i*), we get

$\sigma_s = 1.909 \times 21.88$ N/mm²

= **41.77 N/mm². Ans.**

(*ii*) *Load carried by each bar.*

As $\quad$ load = Stress × Area

$\therefore$ Load carried by steel rod,

$P_s = \sigma_s \times A_s$

$= 41.77 \times 706.86 =$ **29525.5 N. Ans.**

Load carried by copper tube,

$P_c = 45000 - 29525.5$

= **15474.5 N. Ans.**

**Problem 1.20.** *A compound tube consists of a steel tube 140 mm internal diameter and 160 mm external diameter and an outer brass tube 160 mm internal diameter and 180 mm external diameter. The two tubes are of the same length. The compound tube carries an axial load of 900 kN. Find the stresses and the load carried by each tube and the amount it shortens. Length of each tube is 140 mm. Take E for steel as $2 \times 10^5$ N/mm² and for brass as $1 \times 10^5$ N/mm².*

**Sol.** Given :

Internal dia. of steel tube = 140 mm

External dia. of steel tube = 160 mm

$\therefore$ Area of steel tube, $\quad A_s = \frac{\pi}{4}(160^2 - 140^2) = 4712.4$ mm²

Internal dia. of brass tube = 160 mm

External dia. of brass tube = 180 mm

$\therefore$ Area of brass tube, $\quad A_b = \frac{\pi}{4}(180^2 - 160^2) = 5340.7$ mm²

Axial load carried by compound tube,

$P = 900$ kN $= 900 \times 1000 = 900000$ N

Length of each tube, $\quad L = 140$ mm

$E$ for steel, $\quad E_s = 2 \times 10^5$ N/mm²

$E$ for brass, $\quad E_b = 1 \times 10^5$ N/mm²

Let $\quad \sigma_s$ = Stress in steel in N/mm² and

$\sigma_b$ = Stress in brass in N/mm²

Now strain in steel = Strain in brass

$$\frac{\sigma_s}{E_s} = \frac{\sigma_b}{E_b} \qquad \left(\because \text{ Strain} = \frac{\text{Stress}}{E}\right)$$

$$\therefore \quad \sigma_s = \frac{E_s}{E_b} \times \sigma_b = \frac{2 \times 10^5}{1 \times 10^5}\,\sigma_b = 2\sigma_b \qquad \ldots(i)$$

Now load on steel + Load on brass = Total load

or $\sigma_s \times A_s + \sigma_b \times A_b = 900000$ $\quad (\because$ Load = Stress × Area)

or $2\sigma_b \times 4712.4 + \sigma_b \times 5340.7 = 900000$ $\quad (\because \sigma_s = 2\sigma_b)$

or $14765.5\,\sigma_b = 900000$

$$\therefore \quad \sigma_b = \frac{900000}{14765.5} = \textbf{60.95 N/mm}^2. \quad \textbf{Ans.}$$

Substituting the value of $p_b$ in equation (*i*), we get

$$\sigma_s = 2 \times 60.95 = \textbf{121.9 N/mm}^2. \quad \textbf{Ans.}$$

Load carried by brass tube

= Stress × Area

$= \sigma_b \times A_b = 60.95 \times 5340.7$ N

= 325515 N = **325.515 kN. Ans.**

Load carried by steel tube

= 900 − 325.515 = **574.485 kN. Ans.**

Decrease in the length of the compound tube

= Decrease in length of either of the tubes

= Decrease in length of brass tube

= Strain in brass tube × Original length

$$= \frac{\sigma_b}{E_b} \times L = \frac{60.95}{1 \times 10^5} \times 140 = \textbf{0.0853 mm.} \quad \textbf{Ans.}$$

# Method of Sections

## Method of Sections | Analysis of Simple Trusses

**Method of Sections**

In this method, we will cut the truss into two sections by passing a cutting plane through the members whose internal forces we wish to determine. This method permits us to solve directly any member by analyzing the left or the right section of the cutting plane.

To remain each section in equilibrium, the cut members will be replaced by forces equivalent to the internal load transmitted to the members. Each section may constitute of non-concurrent force system from which three equilibrium equations can be written.

$$\Sigma F_H = 0,\ \ \Sigma F_V = 0,\ \ \text{and}\ \ \Sigma M_O = 0$$

Because we can only solve up to three unknowns, it is important not to cut more than three members of the truss. Depending on the type of truss and which members to solve, one may have to repeat Method of Sections more than once to determine all the desired forces.

## Method of Sections Simple Steps:

- Always Start by calculating reactions at supports
- Make a slice through the members you wish to solve
- Treat the half structure as its own static truss
- Solve the truss by taking the sum of forces = 0
- Take the moment about a node of more than one unknown member

## In brief ………..

1. Draw the ***FBD*** for the entire truss system.
2. Determine the ***reactions***. Using the equations of (**2 D**) which states:

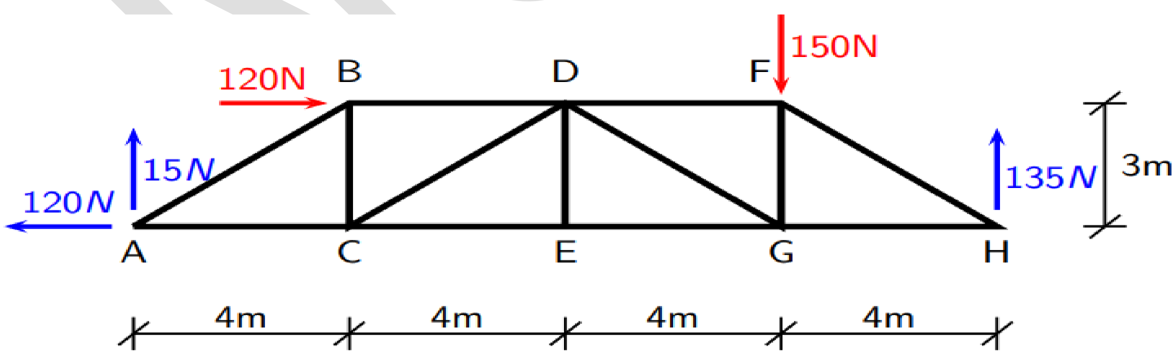
$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0$$

3. Choose the ***section***, and draw ***FBD*** of that *section*, shows how the forces replace the sectioned members.
4. Using the equation of (**2 D**) which states:

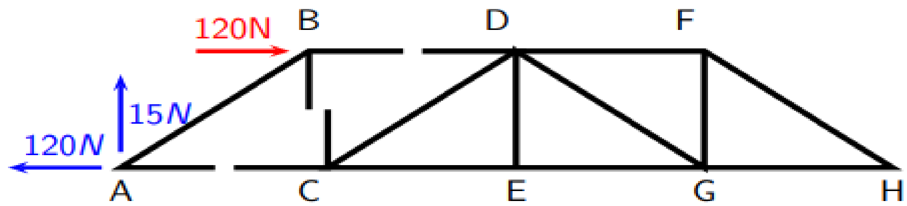
$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0$$

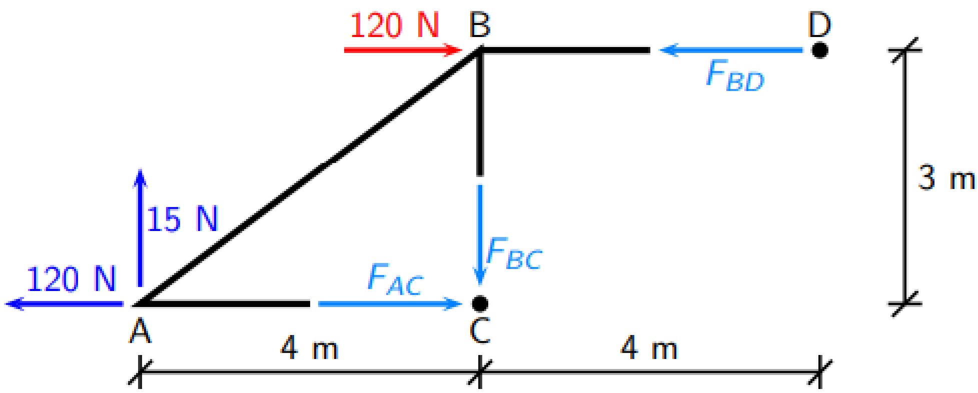
5. The *internal forces* are determined.
6. Choose another *section* or *joint*.

## For example : Analyse the system shown below…



Let's create a section by *cutting* through members AC, BC and BD. Recall that we want to cut through at most three members.




Since $F_{BC}$ is the only force that has a vertical component, it must point down to balance the 15 N force ($A_y$).

Taking moments about point B has both forces at A giving clockwise moments. Therefore, $F_{AC}$ must point to the right to provide a counter-clockwise moment.

Taking moments about point C has the 15 N force acting at A and the 120 N acting at B giving clockwise moments. Therefore, $F_{BD}$ must point to the left to provide a counter-clockwise moment.

$$\uparrow^{+} \sum F_y = +15N - F_{BC} = 0$$
$$F_{BC} = 15N \text{ (tension)}$$

$$\circlearrowleft^{+} \sum M_B = -(120N)(3m) - (15N)(4m) + F_{AC}(3m) = 0$$
$$F_{AC} = \frac{(360 + 60)Nm}{3m} = 140N \text{ (tension)}$$

$$\circlearrowleft^{+} \sum M_C = -(15N)(4m) - (120N)(3m) + F_{BD}(3m) = 0$$
$$F_{BD} = \frac{(60 + 360)Nm}{3m} = 140N \text{ (compression)}$$

**like this we have to analyse the whole problem………**

Method of Sections

~~Procedure~~

This method is used in the analysis of truseses

Procedure

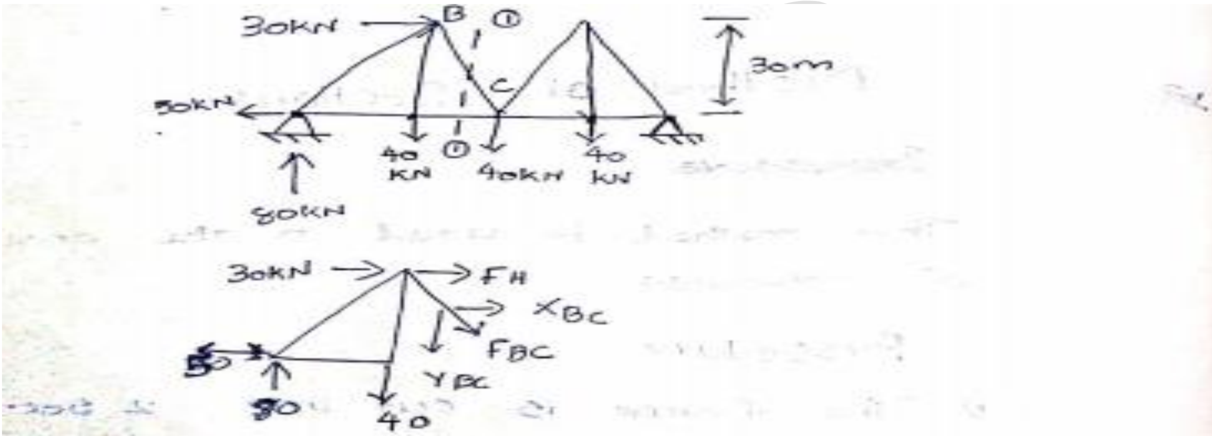
1) The frame is cut into 2 sections

2) each sections is analysed by

using FBD.

3) using equilibrium equation $F_x, F_y = 0$ & $\Sigma M = 0$. The force are determined.

eg:-



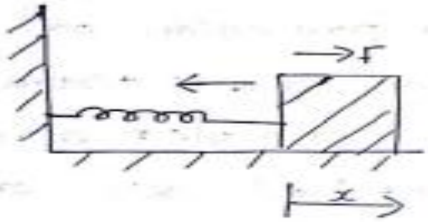
Assumptions

* The truss members are rigidly joined [ie, every member of the truss is in pure compression or pure tension or pure shear stressess]. and the other complex stressess are practically zero.
* load acts on the joints only.
* Weight of trusses member as compared to external load is negligible and not considered for calculation.

Limitations

- It is a lengthy method
- leads to error
- It is very tedious or complex structures
- The section line must not cut more than 3 members. ∴ for a complex structure we have to draw too many FBD.

✓ Hooke's Law




$$[F = -kx]$$

Spring constant

hooke's law of elasticity is an approximation, that states the extension of a spring is indirect proportion with a load applied to it.

Many material obey this law as long as the load doesn't exceed the materials elastic limit.

The material which obey hookes law is linear elastic materials.

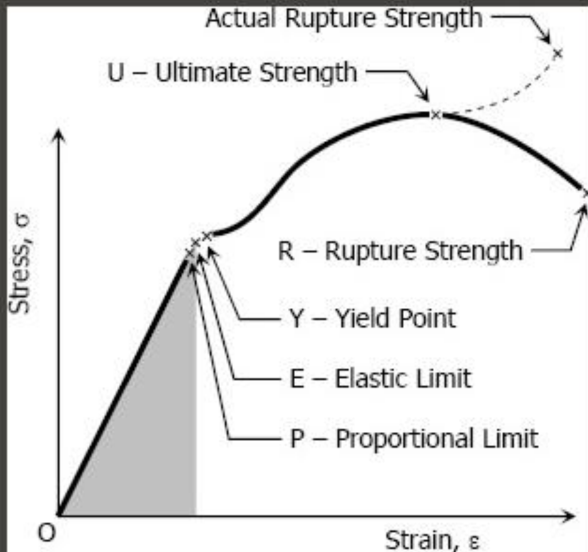
And generally hook's law states that the stress is α strain when the material is loaded within the elastic limit.

# STRESS–STRAIN CURVE

## Stress-strain Diagram

Suppose that a metal specimen be placed in tension-compression-testing machine. As the axial load is gradually increased in increments, the total elongation over the gauge length is measured at each increment of the load and this is continued until failure of the specimen takes place. Knowing the original cross-sectional area and length of the specimen, the normal stress σ and the strain ε can be obtained. The graph of these quantities with the stress σ along the y-axis and the strain ε along the x-axis is called the stress-strain diagram. The stress-strain diagram differs in form for various materials. The diagram shown below is that for a medium-carbon structural steel.

Metallic engineering materials are classified as either ductile or brittle materials. A ductile material is one having relatively large tensile strains up to the point of rupture like structural steel and aluminum, whereas brittle materials has a relatively small strain up to the point of rupture like cast iron and concrete. An arbitrary strain of 0.05 mm/mm is frequently taken as the dividing line between these two classes.




*Stress-strain diagram of a medium-carbon structural steel*

**Proportional Limit (Hooke's Law)**

From the origin O to the point called proportional limit, the stress-strain curve is a straight line. This linear relation between elongation and the axial force causing was first noticed by Sir Robert Hooke in 1678 and is called Hooke's Law that within the proportional limit, the stress is directly proportional to strain or

$$\sigma \propto \varepsilon \quad \text{or} \quad \sigma = k\varepsilon$$

Robert Hooke

The constant of proportionality k is called the Modulus of Elasticity E or Young's Modulus and is equal to the slope of the stress-strain diagram from O to P. Then

$$\sigma = E\varepsilon$$

**Elastic Limit**

The elastic limit is the limit beyond which the material will no longer go back to its original shape when the load is removed, or it is the maximum stress that may e developed such that there is no permanent or residual deformation when the load is entirely removed.

**Elastic and Plastic Ranges**

The region in stress-strain diagram from O to E is called the elastic range. The region from E to R is called the plastic range.

**Yield Point**

Yield point is the point at which the material will have an appreciable elongation or yielding without any increase in load.

**Ultimate Strength**

The maximum ordinate in the stress-strain diagram is the ultimate strength or tensile strength.

**Rapture Strength**

Rapture strength is the strength of the material at rupture. This is also known as the breaking strength.

**Modulus of Resilience**

Modulus of resilience is the work done on a unit volume of material as the force is gradually increased from O to P, in $N \cdot m/m^3$. This may be calculated as the area under the stress-strain curve from the origin O to up to the elastic limit E (the shaded area in the figure). The resilience of the material is its ability to absorb energy without creating a permanent distortion.

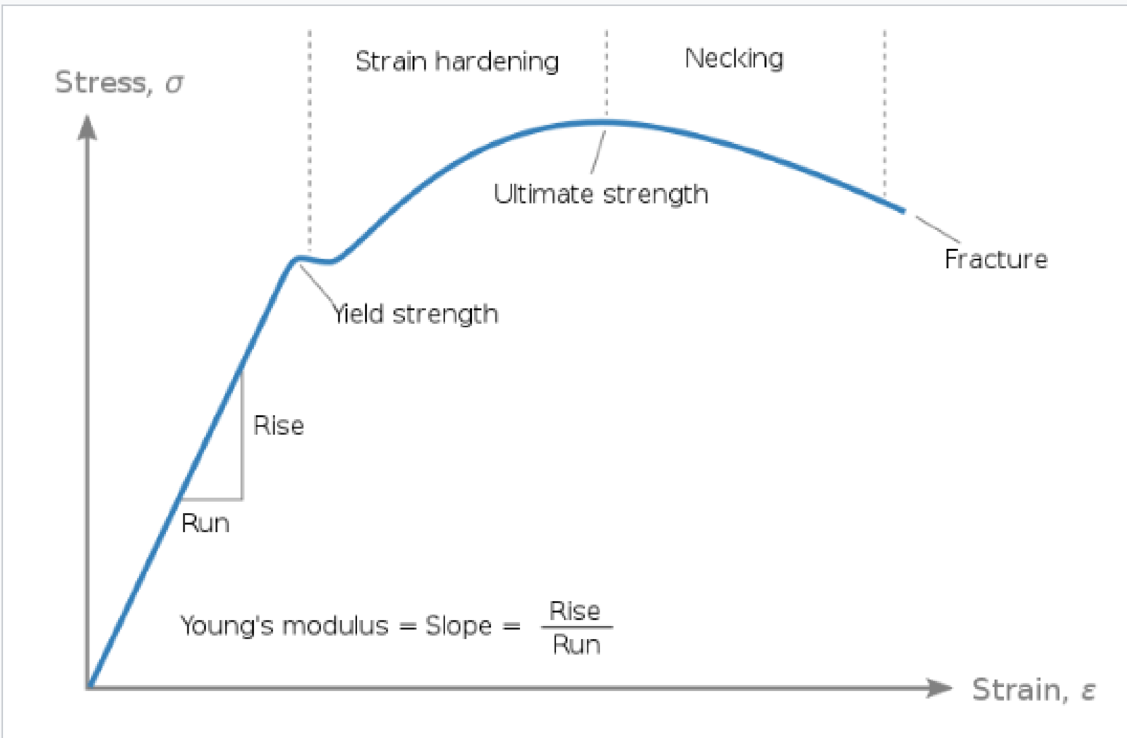
**Modulus of Toughness**

Modulus of toughness is the work done on a unit volume of material as the force is gradually increased from O to R, in $N \cdot m/m^3$. This may be calculated as the area under the entire stress-strain curve (from O to R). The toughness of a material is its ability to absorb energy without causing it to break.

**Working Stress, Allowable Stress, and Factor of Safety**

Working stress is defined as the actual stress of a material under a given loading. The maximum safe stress that a material can carry is termed as the allowable stress. The allowable stress should be limited to values not exceeding the proportional limit. However, since proportional limit is difficult to determine accurately, the allowable tress is taken as either the yield point or ultimate strength divided by a factor of safety. The ratio of this strength (ultimate or yield strength) to allowable strength is called the factor of safety.

## Another detailed graph

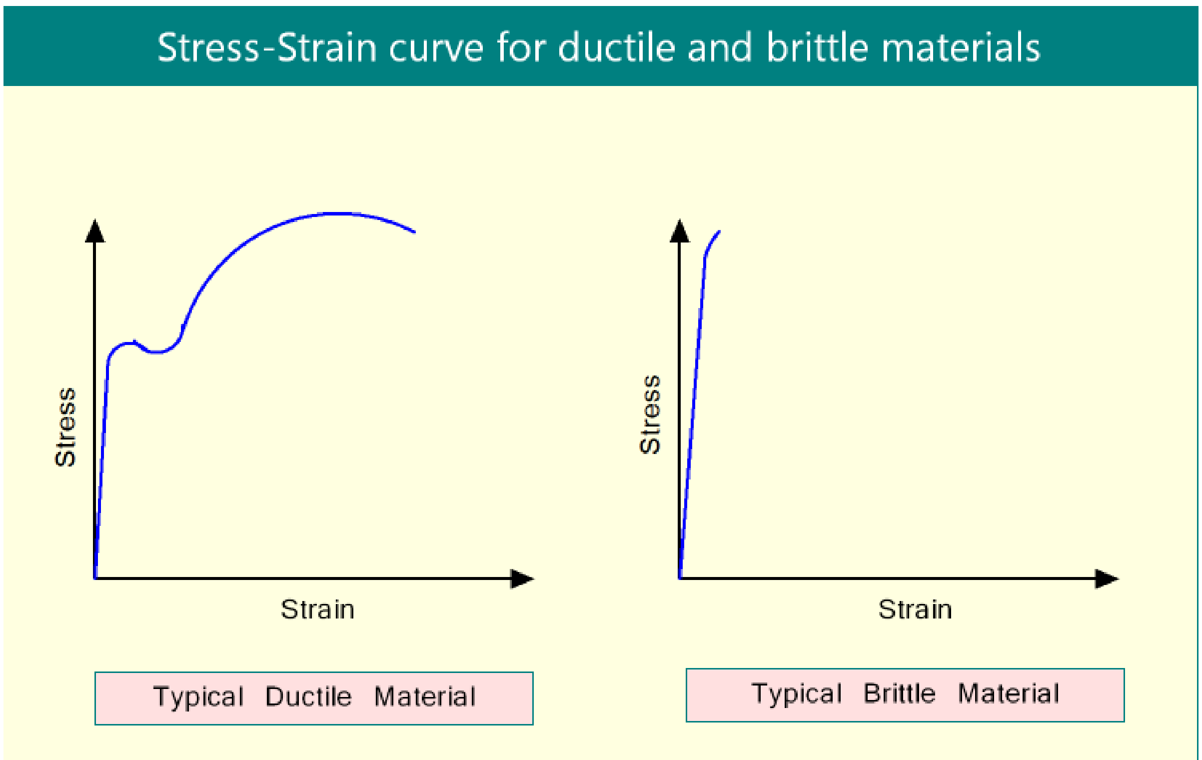



Stress-strain curve typical of a low carbon steel.

## Difference Between Ductile Material and Brittle Material

Every engineering material, when in service, is subjected to external loading of several natures (continuous, repetitive or fluctuating loading). In some applications (for example, metal rolling or bending), it is intended that the component should elongate as much as possible before fracture; while in other applications (for example, stone braking), it is intended that the material should break with minor deformation under external loading. Based on the capability to elongate under external loading, solid materials can be classified in two categories – ductile and brittle.

When external tensile load is applied on a material, initially it undergoes elastic deformation and then plastic deformation starts. An elastic deformation is recoverable, while a plastic deformation is permanent. Ability of a material to exhibit plastic deformation before fracture is the indication of ductility. Materials that show substantial plastic deformation under external loading are called ductile materials; while brittle materials exhibit negligible plastic deformation. Similarities and differences between ductile material and brittle material are provided below.




## Similarities between ductile material and brittle material

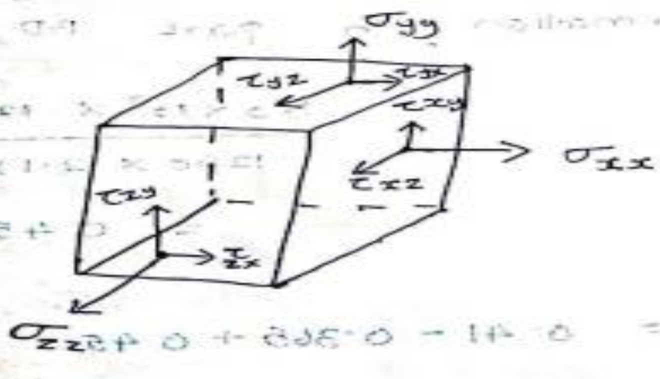
- Both are associated with the plastic deformation of the material under tensile loading.
- Ductility or brittleness is highly temperature dependent. For example, a brittle material can behave like a ductile one at an elevated temperature. Similarly a ductile material at room temperature, when frozen, can automatically convert into brittle material.
- Ductility or brittleness of a material also depends on the inbuilt stress level. Under presence of high residual stress, a ductile material may fail without palpable plastic elongation.

# MODULE 2

## STRAIN ENERGY,STRESS & STRAIN AT A POINT

### Stress At A Point

Stress is a vector quantity and is defined by it's magnitude and direction or it can be defined by the plane on which it is acting. At any point there are infinite no of planes passing through it. and infinite no of vectors acting at the point. All these vectors passing through the point define the state of stress @ a point.



Therefore, the state of stress @ a point;

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

And this called '<u>stress Tensor</u>.'

## <u>Strain @ a point</u>

It is similar to stress tensor. we know that, components of strain.

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix}$$

$\sigma$ = normal stress | $\varepsilon$ = normal strain

$\tau$ = shear " | $\gamma$ = shear ,

## Strain Energy

$$\left[ U = \frac{\sigma^2}{2E} \cdot V \right]$$

V = Volume of the body.

whenever a body is strained the energy is absorbed in the body. The energy which is absorbed in the body due to straining effect is called strain energy.

* **Resilience**

The total strain energy stored in the body. whenever the strainning force is removed the body is capable of doing work. hence resiliance is also defined as capacity of a strained body for doing work on a removal of strain force.

* **proof Resilience**

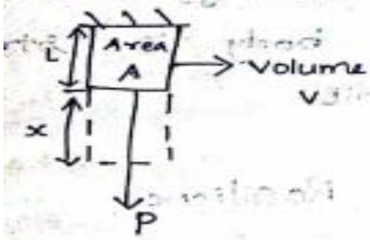
The max strain energy stored in a body. when the body is stressed upto elastic limit.

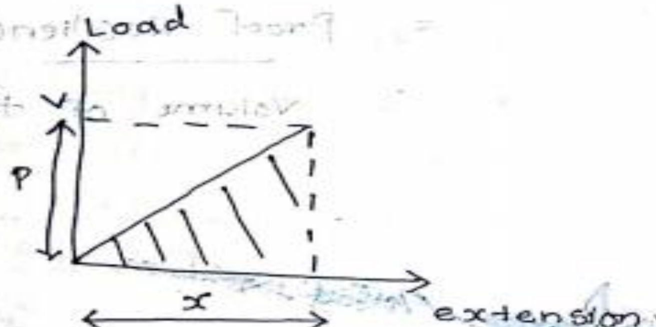
* **Modulus of Resilience**

$$= \frac{\text{Proof Resilience}}{\text{Volume of the body.}}$$

## Derivations for U

### 1) Load is applied gradually




$U =$ Area of triangle.

$$U = \frac{1}{2} \cdot x \cdot P$$

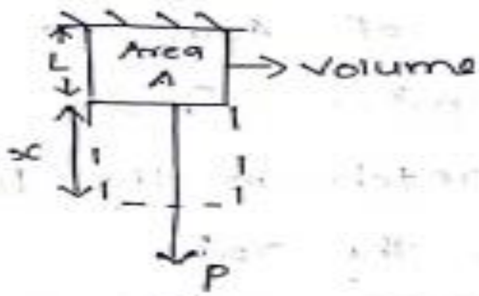
$$U = \frac{1}{2} \cdot eL \cdot \sigma A$$

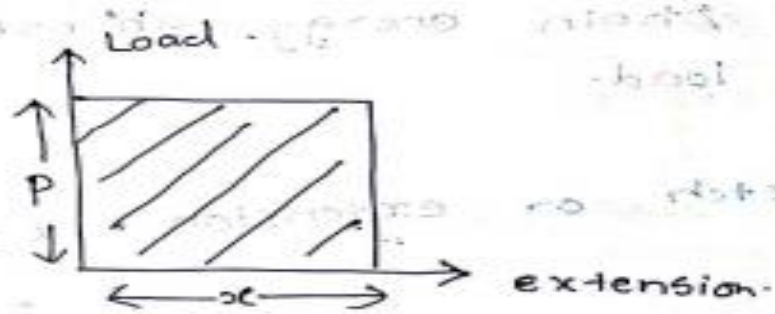
$$\left[ \begin{array}{l} \cdot\ \sigma = \frac{P}{A} \\ \cdot\ e = \frac{x}{L} \\ \cdot\ E = \frac{\sigma}{e} \end{array} \right]$$

$$U = \frac{1}{2} \cdot \frac{\sigma}{E} \cdot L \cdot \sigma A$$

$$\left[ U = \frac{1}{2} \frac{\sigma^2}{E} V \right]$$

2) Load is applied suddenly




U = Area of Rectangle.

$$U = P\,x$$

$$= \sigma_A \cdot eL$$

$$U = \sigma \cdot A \cdot \frac{\sigma}{E} \cdot L$$

$$\therefore \left[ U = \frac{\sigma^2}{E} V \right]$$

**Problem 4.2.** *If in problem 4.1, the tensile load of 60 kN is applied suddenly determine :*

*(i) maximum instantaneous stress induced,*

*(ii) instantaneous elongation in the rod, and*

*(iii) strain energy absorbed in the rod.*

**Sol.** Given :

The data given in problem 4.1 is $d = 40$ mm, Area $= 400\,\pi$ mm$^2$, $L = 5000$ mm, Volume $= 2 \times 10^6\,\pi$ mm$^3$, $E = 2 \times 10^5$ N/mm$^2$ and suddenly applied load, $P = 60000$ N.

*(i) Maximum instantaneous stress induced*

Using equation (4.5),

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{60000}{400\pi} = \mathbf{95.493\ N/mm^2}. \quad \textbf{Ans.}$$

*(ii) Instantaneous elongation in the rod*

Let $x$ = Instantaneous elongation

Then
$$x = \frac{\sigma}{E} \times L = \frac{95.493}{2 \times 10^5} \times 5000 \qquad \text{[see equation (4.1)]}$$
$$= \mathbf{2.38\ mm.} \quad \textbf{Ans.}$$

*(iii) Strain energy is given by,*

$$U = \frac{\sigma^2}{2E} \times V = \frac{95.493^2}{2 \times 2 \times 10^5} \times 2 \times 10^6\,\pi = 143238 \text{ N-mm}$$
$$= \mathbf{143.238\ N\text{-}m.} \quad \textbf{Ans.}$$

**Problem 4.3.** *Calculate instantaneous stress produced in a bar 10 cm² in area and 3 m long by the sudden application of a tensile load of unknown magnitude, if the extension of the bar due to suddenly applied load is 1.5 mm. Also determine the suddenly applied load. Take $E = 2 \times 10^5$ N/mm².*

**Sol.** Given :

Area of bar, $A = 10$ cm$^2 = 1000$ mm$^2$

Length of bar, $L = 3$ m $= 3000$ mm

Extension due to suddenly applied load,

$x = 1.5$ mm

Young's modulus, $E = 2 \times 10^5$ N/mm².

Let $\sigma$ = Instantaneous stress due to sudden load, and

$P$ = Suddenly applied load.

The extension $x$ is given by equation (4.1),

$$x = \frac{\sigma}{E} \times L \quad \text{or} \quad 1.5 = \frac{\sigma}{2 \times 10^5} \times 3000$$

$$\therefore \quad \sigma = \frac{1.5 \times 2 \times 10^5}{3000} = \mathbf{100\ N/mm^2.} \quad \textbf{Ans.}$$

*Suddenly applied load*

The instantaneous stress produced by a sudden load is given by equation (4.5) as

$$\sigma = 2 \times \frac{P}{A} \quad \text{or} \quad 100 = 2 \times \frac{P}{1000}$$

$$\therefore \quad P = \frac{1000 \times 100}{2} = 50000 \text{ N} = \mathbf{50\ kN.} \quad \textbf{Ans.}$$

**Problem 4.4.** *A steel rod is 2 m long and 50 mm in diameter. An axial pull of 100 kN is suddenly applied to the rod. Calculate the instantaneous stress induced and also the instantaneous elongation produced in the rod. Take E = 200 GN/m².*

**Sol.** Given :

Length, $L = 2 \text{ m} = 2 \times 1000 = 2000$ mm

Diameter, $d = 50$ mm

$\therefore$ Area, $A = \frac{\pi}{4} \times 50^2 = 625\,\pi \text{ mm}^2$

Suddenly applied load,

$P = 100 \text{ kN} = 100 \times 1000$ N

Value of $E = 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2$ $\quad (\because G = \text{Giga} = 10^9)$

$= \frac{200 \times 10^9}{10^6} \text{ N/mm}^2 \quad (\because 1 \text{ m} = 1000 \text{ mm} \quad \therefore \text{m}^2 = 10^6 \text{ mm}^2)$

$= 200 \times 10^3 \text{ N/mm}^2$

Using equation (4.5) for suddenly applied load,

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{100 \times 1000}{625\,\pi} \text{ N/mm}^2 = \mathbf{101.86\ N/mm^2.} \quad \textbf{Ans.}$$

Let $dL$ = Elongation

Then
$$dL = \frac{P}{E} \times L = \frac{101.86}{200 \times 10^3} \times 2000 = \mathbf{1.0186\ mm.} \quad \textbf{Ans.}$$

**Problem 4.5.** *A uniform metal bar has a cross-sectional area of 700 mm² and a length of 1.5 m. If the stress at the elastic limit is 160 N/mm², what will be its proof resilience ? Determine also the maximum value of an applied load, which may be suddenly applied without exceeding the elastic limit. Calculate the value of the gradually applied load which will produce the same extension as that produced by the suddenly applied load above.*

*Take E = 2 × 10⁵ N/mm².*

**Sol.** Given :

Area, $A = 700 \text{ mm}^2$

Length, $L = 1.5 \text{ m} = 1500$ mm

$\therefore$ Volume of bar, $V = A \times L = 700 \times 1500 = 1050000 \text{ mm}^3$

Stress at elastic limit, $\sigma^* = 160$ N/mm²

Young's modulus, $E = 2 \times 10^5$ N/mm²

(*i*) Proof resilience is given by equation (4.3), as

$$\text{Proof resilience} = \frac{\sigma^{*2}}{2E} \times \text{Volume} = \frac{160^2}{2 \times 2 \times 10^5} \times 1050000$$

$$= 67200 \text{ N-mm} = \mathbf{67.2 \text{ N-m.}} \quad \textbf{Ans.}$$

(*ii*) Let $P$ = Maximum value of suddenly applied load, and

$P_1$ = Gradually applied load.

Using equation (4.5) for suddenly applied load,

$$\sigma^* = 2 \times \frac{P}{A} \qquad \text{(change } p \text{ to } p^*\text{)}$$

$$\therefore \quad P = \frac{\sigma^* \times A}{2} = \frac{160 \times 700}{2} = 56000 \text{ N} = \mathbf{56 \text{ kN.}} \quad \textbf{Ans.}$$

For gradually applied load,

$$\sigma^* = \frac{P_1}{A}$$

or $$P_1 = \sigma^* \times A = 160 \times 700 = 112000 \text{ N} = \mathbf{112 \text{ kN.}} \quad \textbf{Ans.}$$

**Problem 4.6.** *A tension bar 5 m long is made up of two parts, 3 metre of its length has a cross-sectional area of 10 cm² while the remaining 2 metre has a cross-sectional area of 20 cm². An axial load of 80 kN is gradually applied. Find the total strain energy produced in the bar and compare this value with that obtained in a uniform bar of the same length and having the same volume when under the same load. Take $E = 2 \times 10^5$ N/mm².*

**Sol.** Given :

Total length of bar, $L = 5$ m $= 5000$ mm

Length of 1st part, $L_1 = 3$ m $= 3000$ mm

Area of 1st part, $A_1 = 10$ cm² $= 10 \times 100$ mm² $= 1000$ mm²

∴ Volume of 1st part,

$$V_1 = A_1 \times L_1 = 1000 \times 3000 = 3 \times 10^6 \text{ mm}^3$$

Length of 2nd part, $L_2 = 2$ m $= 2000$ mm

Area of 2nd part, $A_2 = 20$ cm² $= 20 \times 100$ mm² $= 2000$ mm²

∴ Volume of 2nd part, $V_2 = 2000 \times 2000 = 4 \times 10^6$ mm³

Axial gradual load, $P = 80$ kN $= 80 \times 1000 = 80000$ N

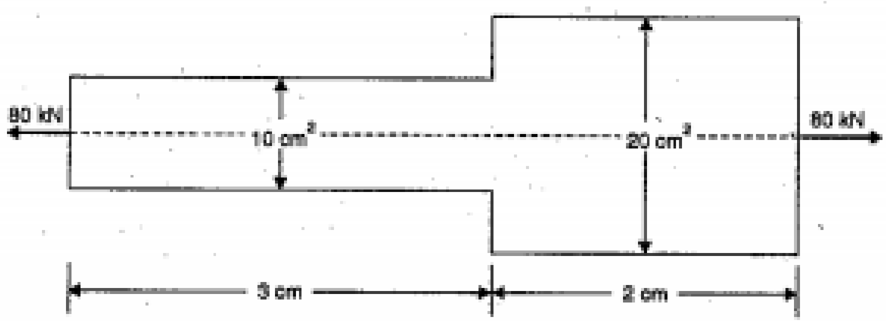



Fig. 4.2

Young's modulus $E = 2 \times 10^5$ N/mm²

Stress in 1st part $\sigma_1 = \frac{\text{Load}}{A_1} = \frac{80000}{1000} = 80$ N/mm²

Stress in 2nd part $\sigma_2 = \frac{P}{A_2} = \frac{80000}{2000} = 40$ N/mm²

Strain energy in 1st part,

$$U_1 = \frac{\sigma_1^2}{2E} \times V_1 = \frac{80^2}{2 \times 2 \times 10^5} \times 3 \times 10^6 = 48000 \text{ N-mm} = 48 \text{ N-m}$$

Strain energy in 2nd part,

$$U_2 = \frac{\sigma_2^2}{2E} \times V_2 = \frac{40^2}{2 \times 2 \times 10^5} \times 4000000 = 16000 \text{ N-mm} = 16 \text{ N-m}$$

∴ Total strain energy produced in the bar,

$$U = U_1 + U_2 = 48 + 16 = \mathbf{64 \text{ N-m.}} \quad \textbf{Ans.}$$

**Strain energy stored in a uniform bar**

Volume of uniform bar, $V = V_1 + V_2 = 3000000 + 4000000 = 7000000$ mm²

Length of uniform bar, $L = 5$ m $= 5000$ mm

Let $A$ = Area of uniform bar

Then $V = A \times L$ or $7000000 = A \times 5000$

∴ $A = \frac{7000000}{5000} = 1400$ mm²

Stress in uniform bar, $\sigma = \frac{P}{A} = \frac{80000}{5000} = 57.143$ N/mm²

∴ Strain energy stored in the uniform bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{57.143^2}{2 \times 2 \times 10^5} \times 7000000$$

$$= 57143 \text{ N-mm} = 57.143 \text{ N-m}$$

∴ $$\frac{\text{Strain energy in the given bar}}{\text{Strain energy in the uniform bar}} = \frac{64}{57.143} = \mathbf{1.12.} \quad \textbf{Ans.}$$

## RELATIONSHIP BETWEEN ELASTIC CONSTANTS

Formula

1) Young's modulus (E) in terms of Bulk (K) modulus & Poisson's (μ) ratio

$$[E = 3K(1 - 2\mu)]$$

2) Relationship b/w mod. of elasticity (E) & Mod. Rigidity. (C)

$$\left[C = \frac{E}{2(1+\mu)}\right] \quad \left[K = \frac{E}{3(1-2\mu)}\right]$$

**Problem 2.8.** *For a material, Young's modulus is given as $1.2 \times 10^5$ N/mm² and Poisson's ratio $\frac{1}{4}$. Calculate the Bulk modulus.*

**Sol.** Given : Young's modulus, $E = 1.2 \times 10^5$ N/mm²

Poisson's ratio, $\mu = \frac{1}{4}$

Let $K$ = Bulk modulus

Using equation (2.10),

$$K = \frac{E}{3(1-2\mu)} = \frac{1.2 \times 10^5}{3\left(1 - \frac{2}{4}\right)} = \frac{1.2 \times 10^5}{3 \times \frac{1}{2}}$$

$$= \frac{2 \times 1.2 \times 10^5}{3} = \mathbf{0.8 \times 10^5\ N/mm^2.} \quad \textbf{Ans.}$$

**Problem 2.9.** *A bar of 30 mm diameter is subjected to a pull of 60 kN. The measured extension on gauge length of 200 mm is 0.1 mm and change in diameter is 0.004 mm. Calculate :*

*(i) Young's modulus* *(ii) Poisson's ratio and*

*(iii) Bulk modulus.*

**Sol.** Given : Dia. of bar, $d = 30$ mm

∴ Area of bar, $A = \frac{\pi}{4}(30)^2 = 225\pi$ mm²

Pull, $P = 60$ kN $= 60 \times 1000$ N

Gauge length, $L = 200$ mm

Extension, $\delta L = 0.1$ mm

Change in dia., $\delta d = 0.004$ mm

*(i) Young's modulus (E)*

Tensile stress, $\sigma = \frac{P}{A} = \frac{60000}{225\pi} = 84.87$ N/mm²

Longitudinal strain $= \frac{\delta L}{L} = \frac{0.1}{200} = 0.0005$

∴ Young's modulus, $E = \frac{\text{Tensile stress}}{\text{Longitudinal strain}}$

$$= \frac{84.87}{0.0005} = 16.975 \times 10^4 \text{ N/mm}^2$$

$$= \mathbf{1.6975 \times 10^5\ N/mm^2.} \quad \textbf{Ans.}$$

*(ii) Poisson's ratio (μ)*

Poisson's ratio is given by equation (2.3) as

Poisson's ratio $(\mu) = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$

$$= \frac{\left(\frac{\delta d}{d}\right)}{0.0005} \qquad \left(\because \text{ Lateral strain} = \frac{\delta L}{d}\right)$$

$$= \frac{\left(\frac{0.004}{30}\right)}{0.0005} = \frac{0.000133}{0.0005} = \mathbf{0.266.} \quad \textbf{Ans.}$$

*(iii) Bulk modulus (K)*

Using equation (2.10), we get

$$K = \frac{E}{3(1-2\mu)} = \frac{1.6975 \times 10^5}{3(1-0.266 \times 2)}$$

$$= \mathbf{1.209 \times 10^5\ N/mm^2.} \quad \textbf{Ans.}$$

**Problem 2.10.** *Determine the Poisson's ratio and bulk modulus of a material, for which Young's modulus is 1.2 × 10⁵ N/mm² and modulus of rigidity is 4.8 × 10⁴ N/mm².*

**Sol.** Given :

Young's modulus, $E = 1.2 \times 10^5$ N/mm²

Modulus of rigidity, $C = 4.8 \times 10^4$ N/mm²

Let the Poisson's ratio = $\mu$

Using equation (2.16), we get

$$E = 2C(1 + \mu)$$

or $$1.2 \times 10^5 = 2 \times 4.8 \times 10^4 (1 + \mu)$$

or $$(1 + \mu) = \frac{1.2 \times 10^5}{2 \times 4.8 \times 10^4} = 1.25 \quad \text{or} \quad \mu = 1.25 - 1.0 = \mathbf{0.25.} \quad \textbf{Ans.}$$

Bulk modulus is given by equation (2.10) as

$$K = \frac{E}{3(1-2\mu)} = \frac{1.2 \times 10^5}{3(1-0.25 \times 2)} \qquad (\because \mu = 0.25)$$

$$= \mathbf{8 \times 10^4\ N/mm^2.} \quad \textbf{Ans.}$$

**Problem 2.11.** *A bar of cross-section 8 mm × 8 mm is subjected to an axial pull of 7000 N. The lateral dimension of the bar is found to be changed to 7.9985 mm × 7.9985 mm. If the modulus of rigidity of the material is 0.8 × 10⁵ N/mm², determine the Poisson's ratio and modulus of elasticity.*

**Sol.** Given :

Area of section = 8 × 8 = 64 mm²

Axial pull, $P = 7000$ N

Lateral dimensions = 7.9985 mm × 7.9985 mm

Volume of $C = 0.8 \times 10^5$ N/mm²

Let $\mu$ = Poisson's ratio and

$E$ = Modulus of elasticity.

Now lateral strain = $\dfrac{\text{Change in lateral dimension}}{\text{Original lateral dimension}}$

$$= \frac{8 - 7.9985}{8} = \frac{0.0015}{8} = 0.0001875.$$

To find the value of Poisson's ratio, we must know the value of longitudinal strain. But in this problem, the length of bar and the axial extension is not given. Hence longitudinal strain cannot be calculated. But axial stress can be calculated. Then longitudinal, strain will be equal to axial stress divided by $E$.

$\therefore$ Axial stress, $\sigma = \dfrac{P}{\text{Area}} = \dfrac{7000}{64} = 109.375$ N/mm$^2$ and longitudinal strain = $\dfrac{\sigma}{E}$

But lateral strain = $\mu$ × longitudinal strain = $\mu \times \dfrac{\sigma}{E}$

or $$0.0001875 = \frac{\mu \times 109.375}{E} \quad (\because \text{ Lateral strain} = 0.0001875)$$

$\therefore$ $$\frac{E}{\mu} = \frac{109.375}{0.0001875} = 583333.33$$

or $$E = 583333.33\mu \quad \text{...}(i)$$

Using equation (2.17), we get

$$C = \frac{E}{2(1+\mu)} \quad \text{or} \quad E = 2C(1 + \mu)$$

$$= 2 \times 0.8 \times 10^5 (1 + \mu) \quad (\because \ C = 0.8 \times 10^5)$$

or $$583333.33\mu = 2 \times 0.8 \times 10^5 (1 + \mu) \quad (\because \ E = 583333.33\mu)$$

or $$1 + \mu = \frac{583333.33\mu}{2 \times 0.8 \times 10^5} = 3.6458\mu$$

$\therefore$ $$1 = 3.6458\mu - \mu = 2.6458\mu$$

$\therefore$ Poisson's ratio = $\mu = \dfrac{1}{2.6458} =$ **0.378. Ans.**

Modulus of elasticity ($E$) is obtained by substituting the value of $\mu$ in equation ($i$).

$\therefore$ $$E = 583333.33\mu$$

$\therefore$ $$E = \frac{583333.33}{2.6458} = \mathbf{2.2047 \times 10^5\ N/mm^2.}$$ **Ans.**

**Problem 2.12.** *Calculate the modulus of rigidity and bulk modulus of a cylindrical bar of diameter 30 mm and of length 1.5 m if the longitudinal strain in a bar during a tensile stress is four times the lateral strain. Find the change in volume, when the bar is subjected to a hydrostatic pressure of 100 N/mm². Take E = 1 × 10⁵ N/mm².*

**Sol.** Given :

Dia. of bar, $d = 30$ mm

Length of bar, $L = 1.5$ m $= 1.5 \times 1000 = 1500$ mm

$\therefore$ Volume of bar, $V = \dfrac{\pi}{4} d^2 \times L = \dfrac{\pi}{4} \times 30 \times 1500$

$= 1060287.52$ mm$^3$

Longitudinal strain = 4 × Lateral strain

Hydrostatic pressure, $p = 100\ \text{N/mm}^2$

$$\therefore \quad \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{1}{4} = 0.25$$

or Poisson's ratio, $\mu = 0.25$

Let $C$ = Modulus of rigidity

$K$ = Bulk modulus

$E$ = Young's modulus = $1 \times 10^5\ \text{N/mm}^2$

Using equation (2.16), we get

$$E = 2C(1 + \mu)$$

or

$$1 \times 10^5 = 2C(1 + 0.25)$$

$$\therefore \quad C = \frac{1 \times 10^5}{2 \times 1.25} = 4 \times 10^4\ \text{N/mm}^2. \quad \textbf{Ans.}$$

For bulk modulus, using equation (2.11), we get

$$E = 3K(1 - 2\mu)$$

or

$$1 \times 10^5 = 3K(1 - 2 \times 0.25) \qquad (\because \mu = 0.25)$$

$$\therefore \quad K = \frac{1 \times 10^5}{3 \times 0.5} = 0.667 \times 10^5\ \text{N/mm}^2. \quad \textbf{Ans.}$$

Now using equation (2.9), we get

$$K = \frac{p}{\text{Volumetric strain}} = \frac{p}{\left(\frac{dV}{V}\right)}$$

where $p = 100\ \text{N/mm}^2$

$$\therefore \quad 0.667 \times 10^5 = \frac{100}{\left(\frac{dV}{V}\right)}$$

or

$$\frac{dV}{V} = \frac{100}{0.667 \times 10^5} = 1.5 \times 10^{-3}$$

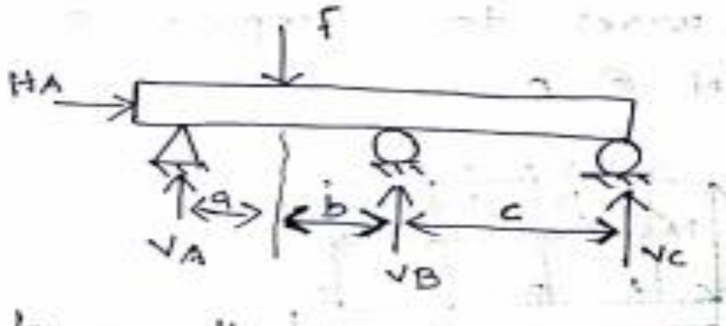
$$\therefore \quad dV = V \times 1.5 \times 10^{-3} = 1060287.52 \times 1.5 \times 10^{-3}$$

$$= 1590.43\ \text{mm}^3. \quad \textbf{Ans.}$$

# Statically Indeterminate Structure

In statics a structure is hyper statics [Statically Indeterminate] when the static equilibrium equations are in Suffcient for determining the internal forces and reactions on that structure.

eg:-



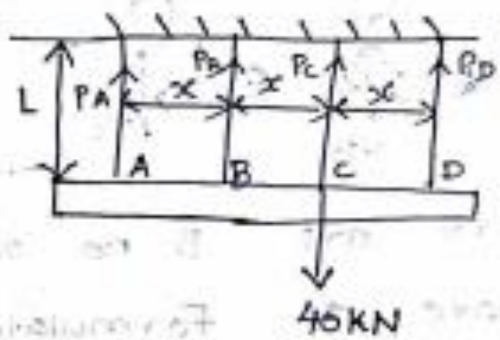

here, there are 5 no of unknowns so, we have to formulate 5 set of equations. we know the static equilibrium equation ie,

$f_x = f_y = 0$

$\Sigma M = 0$

∴ we can create only 3 set's equation and the additional 2 set's of equation are create by considering nature of determination.

4 identical wires support a rigid bar is shown in fig. if the stress is not to exceed 150 N/mm². Find the minimum required diameter of wires to support a load of 40 KN @ C.




Applying equilibrium equation

① $\Sigma F_y = 0$

$$\Rightarrow 40 - P_A - P_B - P_C - P_D \quad - ①$$

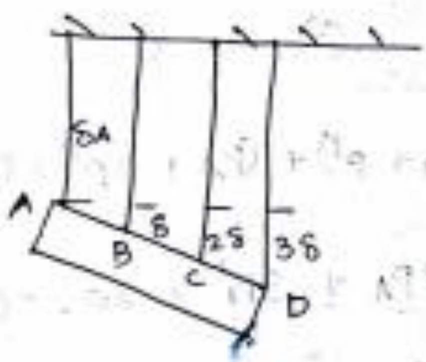
$$P_A + P_B + P_C + P_D = 40$$

② $\Sigma M_A = 0$

$40\,KN = P_B x + P_C 2x + P_D 3x - 40 \times 2x$

$\Rightarrow P_B + 2P_C + 3P_D = 80$ - ②

The 2 equation are not enough to find the 4 unknown. So we have to consider deformed shape of the bar.




Assume $\delta$ be the elongation of wire B when compared to wire A.

$\delta_B = \delta_A + \delta$

$\delta_C = \delta_A + 2\delta$

$\delta_D = \delta_A + 3\delta$

Let $p'$ be the force required for their elongation $\delta$, then $\delta_A$ is the normal & deflection corresponding to the force $P_A$. ie,

$$\left.\begin{aligned} P_A &= \delta_A \\ P_B &= P_A + p' \\ P_C &= P_A + 2p' \\ P_D &= P_A + 3p' \end{aligned}\right\} \quad (3)$$

Substitute (3) in (1)

$$P_A + (P_A + p') + (P_A + 2p') + (P_A + 3p') = 40$$

$$2P_A + 3P' = 20 \quad -(4)$$

(3) in (2)

$$P_A + P' + 2(P_A + 2P') + 3(P_A + 3P') = 80$$

$$3P_A + 7P' = 40 \quad -(5)$$

By solving (4) & (5)

$$\begin{bmatrix} P_A = 4\,kN \\ P' = 4\,kN \end{bmatrix}$$

By substitute

$P_B = 8\,kN$

$P_C = 12\,kN$

$P_D = 16\,kN$

$$\sigma = \frac{P}{A}$$

$$150 = \frac{16 \times 10^3}{\frac{\pi}{4} d^2}$$

$$d = 11\,mm$$

# MODULE 3

# TORSION

A shaft is said to be in torsion, when equal and opposite torques are applied at the two ends of the shaft. The torque is equal to the product of the force applied (tangentially to the ends of a shaft) and radius of the shaft. Due to the application of the torques at the two ends, the shaft is subjected to a twisting moment. This causes the shear stresses and shear strains in the material of the shaft.

## 16.2. DERIVATION OF SHEAR STRESS PRODUCED IN A CIRCULAR SHAFT SUBJECTED TO TORSION

When a circular shaft is subjected to torsion, shear stresses are set up in the material of the shaft. To determine the magnitude of shear stress at any point on the shaft, consider a shaft fixed at one end *AA* and free at the end *BB* as shown in Fig. 16.1. Let *CD* is any line on the outer surface of the shaft. Now let the shaft is subjected to a torque *T* at the end *BB* as shown in Fig. 16.2. As a result of this torque *T*, the shaft at the end *BB* will rotate clockwise and every cross-section of the shaft will be subjected to shear stresses. The point *D* will shift to *D'* and hence line *CD* will be deflected to *CD'* as shown in Fig. 16.2 (*a*). The line *OD* will be shifted to *OD'* as shown in Fig. 16.2 (*b*).

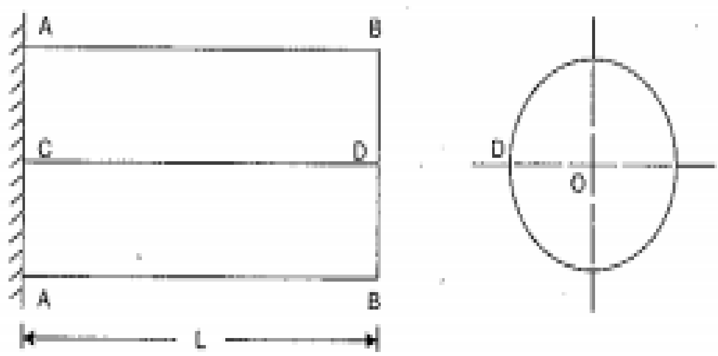
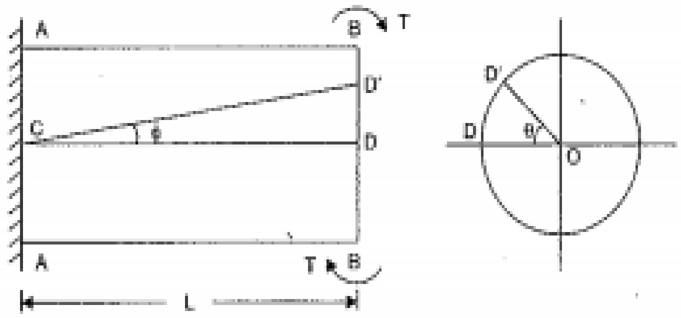



Fig. 16.1. Shaft fixed at one end *AA* before torque *T* is applied.

Let $R$ = Radius of shaft
$L$ = Length of shaft
$T$ = Torque applied at the end *BB*
$\tau$ = Shear stress induced at the surface of the shaft due to torque $T$
$C$ = Modulus of rigidity of the material of the shaft

$\phi = \angle DCD'$ also equal to shear strain
$\theta = \angle DOD'$ and is also called angle of twist.




(a) (b)

Fig. 16.2. Shaft fixed at AA and subjected to torque T at BB.

Now distortion at the outer surface due to torque $T$

$$= DD'$$

$\therefore$ Shear strain at outer surface

$$= \text{Distortion per unit length}$$

$$= \frac{\text{Distortion at the outer surface}}{\text{Length of shaft}} = \frac{DD'}{L}$$

$$= \frac{DD'}{CD} = \tan\phi$$

$$= \phi \qquad (\text{if } \phi \text{ is very small then } \tan\phi \simeq \phi)$$

$\therefore$ Shear strain at outer surface,

$$\phi = \frac{DD'}{L} \qquad ...(i)$$

Now from Fig. 16.2 (b).

Arc $DD' = OD \times \theta = R\theta$ $\qquad$ ($\because$ $OD = R$ = Radius of shaft)

Substituting the value of $DD'$ in equation (i), we get

Shear strain at outer surface

$$\phi = \frac{R \times \theta}{L} \qquad ...(ii)$$

Now the modulus of rigidity ($C$) of the material of the shaft is given as

$$C = \frac{\text{Shear stress induced}}{\text{Shear strain produced}} = \frac{\text{Shear stress at the outer surface}}{\text{Shear strain at outer surface}}$$

$$= \frac{\tau}{\left(\frac{R\theta}{L}\right)} \qquad \left(\because \text{ From equation (ii), shear strain} = \frac{R\theta}{L}\right)$$

$$= \frac{\tau \times L}{R\theta}$$

$$\therefore \qquad \frac{C\theta}{L} = \frac{\tau}{R} \qquad ...(16.1)$$

$$\therefore \qquad \tau = \frac{R \times C \times \theta}{L}$$

Now for a given shaft subjected to a given torque ($T$), the values of $C$, $\theta$ and $L$ are constant. Hence shear stress produced is proportional to the radius $R$.

$$\therefore \qquad \tau \propto R \quad \text{or} \quad \frac{\tau}{R} = \text{constant} \qquad \ldots(iii)$$

If $q$ is the shear stress induced at a radius '$r$' from the centre of the shaft then

$$\frac{\tau}{R} = \frac{q}{r} \qquad \ldots(16.2)$$

But $\qquad \dfrac{\tau}{R} = \dfrac{C\theta}{L}$ from equation (16.1)

$$\therefore \qquad \frac{\tau}{R} = \frac{C\theta}{L} = \frac{q}{r} \qquad \ldots(16.3)$$

From equation ($iii$), it is clear that shear stress at any point in the shaft is proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the outer surface and shear stress is zero at the axis of the shaft.

**16.2.1. Assumptions Made in the Derivation of Shear Stress Produced in a Circular Shaft Subjected to Torsion.** The derivation of shear stress produced in a circular shaft subjected to torsion, is based on the following assumptions :

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. The shaft is of uniform circular section throughout.
4. Cross-sections of the shaft, which are plane before twist remain plain after twist.
5. All radii which are straight before twist remain straight after twist.

## 16.3. MAXIMUM TORQUE TRANSMITTED BY A CIRCULAR SOLID SHAFT

The maximum torque transmitted by a circular solid shaft, is obtained from the maximum shear stress induced at the outer surface of the solid shaft. Consider a shaft subjected to a torque $T$ as shown in Fig. 16.3.

Let $\quad \tau$ = Maximum shear stress induced at the outer surface

$R$ = Radius of the shaft

$q$ = Shear stress at a radius '$r$' from the centre.

Consider an elementary circular ring of thickness '$dr$' at a distance '$r$' from the centre as shown in Fig. 16.3. Then the area of the ring,

$$dA = 2\pi r dr$$

From equation (16.2), we have

$$\frac{\tau}{R} = \frac{q}{r}$$

$\therefore$ Shear stress at the radius $r$,

$$q = \frac{\tau}{R}\, r = \tau \frac{r}{R}$$

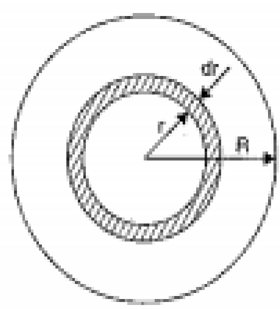



Fig. 16.3

∴ Turning force on the elementary circular ring

$$= \text{Shear stress acting on the ring} \times \text{Area of ring}$$

$$= q \times dA$$

$$= \tau \times \frac{r}{R} \times 2\pi r dr \qquad \left(\because \; q = \tau \times \frac{r}{R}\right)$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr$$

Now turning moment due to the turning force on the elementary ring,

$$dT = \text{Turning force on the ring} \times \text{Distance of the ring from the axis}$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr \times r$$

$$= \frac{\tau}{R} \times 2\pi r^3 dr \qquad \text{...[16.3 (A)]}$$

∴ The total turning moment (or total torque) is obtained by integrating the above equation between the limits $O$ and $R$

$$\therefore \qquad T = \int_0^R dT = \int_0^R \frac{\tau}{R} \times 2\pi r^3 dr$$

$$= \frac{\tau}{R} \times 2\pi \int_0^R r^3 dr = \frac{\tau}{R} \times 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\tau}{R} \times 2\pi \times \frac{R^4}{4} = \tau \times \frac{\pi}{2} \times R^3$$

$$= \tau \times \frac{\pi}{2} \times \left(\frac{D}{2}\right)^3 \qquad \left(\because \; R = \frac{D}{2}\right)$$

$$= \tau \times \frac{\pi}{2} \times \frac{D^3}{8} = \tau \times \frac{\pi D^3}{16} = \frac{\pi}{16} \tau D^3 \qquad \text{...(16.4)}$$

**Problem 16.1.** *A solid shaft of 150 mm diameter is used to transmit torque. Find the maximum torque transmitted by the shaft if the maximum shear stress induced to the shaft is 45 $N/mm^2$.*

**Sol.** Given :

Diameter of the shaft, $D = 150$ mm

Maximum shear stress, $\tau = 45$ N/mm$^2$

Let $T$ = Maximum torque transmitted by the shaft.

Using equation (16.4), $T = \frac{\pi}{16} \tau D^3 = \frac{\pi}{16} \times 45 \times 150^3$

$= 29820586$ N-mm $=$ **29820.586 N-m. Ans.**

**Problem 16.2.** *The shearing stress is a solid shaft is not to exceed 40 $N/mm^2$ when the torque transmitted is 20000 N-m. Determine the minimum diameter of the shaft.*

**Sol.** Given :

Maximum shear stress, $\tau = 40$ N/mm$^2$

Torque transmitted, $T = 20000$ N-m $= 20000 \times 10^3$ N-mm

Let $D$ = Minimum diameter of the shaft in mm.

Using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or

$$D = \left(\frac{16T}{\pi\tau}\right)^{1/3} = \left(\frac{16 \times 20000 \times 10^3}{\pi \times 40}\right)^{1/3} = \mathbf{136.2\ mm.}\quad \textbf{Ans.}$$

## 16.4. TORQUE TRANSMITTED BY A HOLLOW CIRCULAR SHAFTS

Torque transmitted by a hollow circular shaft is obtained in the same way as for a solid shaft. Consider a hollow shaft. Let it is subjected to a torque $T$ as shown in Fig. 16.4. Take an elementary circular ring of thickness '$dr$' at a distance $r$ from the centre as shown in Fig. 16.4.

Let $R_0$ = Outer radius of the shaft

$R_i$ = Inner radius of the shaft

$r$ = Radius of elementary circular ring

$dr$ = Thickness of the ring

$\tau$ = Maximum shear stress induced at outer surface of the shaft

$q$ = Shear stress induced on the elementary ring

$dA$ = Area of the elementary circular ring

$= 2\pi r \times dr$

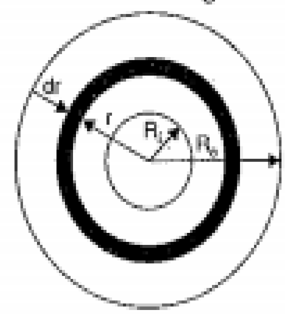


Fig. 16.4. Hollow shaft.

Shear stress at the elementary ring is obtained from equation (16.2) as

$$\frac{\tau}{R_0} = \frac{q}{r} \qquad (\because \text{ Here outer radius } R = R_0)$$

$$\therefore \quad q = \frac{\tau}{R_0} \times r$$

$\therefore$ Turning force on the ring = Stress × Area = $q \times dA$

$$= \frac{\tau}{R_0} r \times 2\pi r dr \qquad \left(\because \ q = \frac{\tau}{R_0} r\right)$$

$$= 2\pi \frac{\tau}{R_0} r^2\, dr$$

Turning moment ($dT$) on the ring,

$dT$ = Turning force × Distance of the ring from centre

$$= 2\pi \frac{\tau}{R_0} r^2 dr \times r = 2\pi \frac{\tau}{R_0} r^3\, dr$$

The total turning moment (or total torque $T$) is obtained by integrating the above equation between the limits $R_i$ and $R_0$.

$$\therefore \quad T = \int_{R_i}^{R_0} dT = \int_{R_i}^{R_0} 2\pi \frac{\tau}{R_0} r^3\, dr$$

$$= 2\pi \frac{\tau}{R_0} \int_{R_i}^{R_0} r^3\, dr$$

($\because$ $\tau$ and $R_0$ are constant and can be taken outside the integral)

$$= 2\pi \frac{\tau}{R_0}\left[\frac{r^4}{4}\right]_{R_i}^{R_0} = 2\pi \frac{\tau}{R_0}\left[\frac{R_0^4 - R_i^4}{4}\right]$$

$$= \frac{\pi}{2}\tau\left[\frac{R_0^4 - R_i^4}{R_0}\right] \quad ...(16.5)$$

Let $D_0$ = Outer diameter of the shaft

$D_i$ = Inner diameter of the shaft.

Then $R_0 = \frac{D_0}{2}$ and $R_i = \frac{D_i}{2}$.

Substituting the values of $R_0$ and $R_i$ in equation (16.5),

$$T = \frac{\pi}{2}\tau\left[\frac{\left(\frac{D_0}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4}{\left(\frac{D_0}{2}\right)}\right] = \frac{\pi}{2}\tau\left[\frac{\frac{D_0^4}{16} - \frac{D_i^4}{16}}{\frac{D_0}{2}}\right]$$

$$= \frac{\pi}{2}\tau\left[\frac{D_0^4 - D_i^4}{16} \times \frac{2}{D_0}\right]$$

$$= \frac{\pi}{16}\tau\left[\frac{D_0^4 - D_i^4}{D_0}\right] \quad ...(16.6)$$

## 16.5. POWER TRANSMITTED BY SHAFTS

Once the expression for torque ($T$) for a solid or a hollow shaft is obtained, power transmitted by the shafts can be determined.

Let $N$ = r.p.m. of the shaft

$T$ = Mean torque transmitted in N-m

$\omega$ = Angular speed of shaft.

Then $\text{Power} = \frac{2\pi NT^*}{60}$ watts ...(16.7)

$= \omega \times T$ $\left(\because \frac{2\pi N}{60} = \omega\right)$

$= T \times \omega$ ...[16.7 (A)]

**Problem 16.3.** *In a hollow circular shaft of outer and inner diameters of 20 cm and 10 cm respectively, the shear stress is not to exceed 40 N/mm². Find the maximum torque which the shaft can safely transmit.*

**Sol.** Given :

Outer diameter, $D_0$ = 20 cm = 200 mm

Inner diameter, $D_i$ = 10 cm = 100 mm

Maximum shear stress, $\tau$ = 40 N/mm²

Let $T$ = Maximum torque transmitted by the shaft.

Using equation (16.6),

$$T = \frac{\pi}{16}\tau\left[\frac{D_0^4 - D_i^4}{D_0}\right] = \frac{\pi}{16} \times 40\left[\frac{200^4 - 100^4}{200}\right]$$

$$= \frac{\pi}{16} \times 40\left[\frac{16 \times 10^8 - 1 \times 10^8}{200}\right] = 58904860 \text{ Nmm}$$

$$= \mathbf{58904.86 \text{ Nm.}} \quad \textbf{Ans.}$$

**Problem 16.4.** *Two shafts of the same material and of same lengths are subjected to the same torque, if the first shaft is of a solid circular section and the second shaft is of hollow circular section, whose internal diameter is 2/3 of the outside diameter and the maximum shear stress developed in each shaft is the same, compare the weights of the shafts.*

(AMIE, Summer 1989)

$0.4573\ D_0$

**Problem 16.6.** *A hollow circular shaft 20 mm thick transmits 300 kW power at 200 r.p.m. Determine the external diameter of the shaft if the shear strain due to torsion is not to exceed 0.00086. Take modulus of rigidity = 0.8 × 10⁵ N/mm².*

(AMIE, Summer 1989 Converted to S.I. units)

**Sol.** Given :

Thickness, $t = 20$ mm

Power transmitted, $P = 300$ kW $= 300{,}000$ W

Speed, $N = 200$ r.p.m.

Shear strain, $\phi = 0.00086$

Modulus of rigidity, $C = 0.8 \times 10^5$ N/mm²

Let $D_0$ = External dia. of shaft and

$D_i$ = Internal dia. of shaft

Then $D_0 = D_i + 2t = D_i + 2 \times 20$

$$\therefore \quad D_i = D_0 - 40 \qquad \ldots(i)$$

Using equation (16.7),

$$P = \frac{2\pi NT}{60} \quad (\text{or } T \times \omega) \quad \text{or} \quad 300000 = \frac{2\pi \times 200 \times T}{60} \quad (\text{or } T \times \omega)$$

$$\therefore \quad T = \frac{300000 \times 60}{2\pi \times 200} = 14323.9 \text{ Nm}$$

$$= 14323.9 \times 1000 \text{ Nmm} = 14323900 \text{ Nmm.}$$

Also we know, $C = \dfrac{\text{Shear stress}}{\text{Shear strain}}$

or $$0.8 \times 10^5 = \frac{\text{Shear stress}}{0.00086}$$

$\therefore$ Shear stress $(\tau) = 0.8 \times 10^5 \times 0.00086 = 68.8$ N/mm²

Now using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

or $$14323900 = \frac{\pi}{16} \times 68.8 \times \frac{(D_0^4 - D_i^4)}{D_0}$$

or $$\frac{14323900 \times 16 \times D_0}{\pi \times 68.8} = D_0^4 - D_i^4$$

$$1060334.6\, D_0 = D_0^4 - D_i^4 = (D_0^2 + D_i^2)(D_0^2 - D_i^2)$$

Substituting the value of $D_i$ from equation $(i)$ into the above equation, we get

$$1060334.6\, D_0 = [D_0^2 + (D_0 - 40)^2][D_0^2 - (D_0 - 40)^2]$$
$$= [D_0^2 + D_0^2 + 1600 - 80D_0][D_0^2 - D_0^2 - 1600 + 80D_0]$$
$$= (2D_0^2 + 1600 - 80D_0)(80D_0 - 1600)$$
$$= 2(D_0^2 + 800 - 40D_0)80(D_0 - 20)$$
$$= 160(D_0^2 - 40D_0 + 800)(D_0 - 20)$$

or $$\frac{1060334.6 D_0}{160} = (D_0^2 - 40D_0 + 800)(D_0 - 20)$$

or $$6627\, D_0 = D_0^3 - 20D_0^2 - 40D_0^2 + 800D_0 + 800D_0 - 16000$$
$$= D_0^3 - 60D_0^2 + 1600D_0 - 16000$$

or $$D_0^3 - 60D_0^2 + 1600D_0 - 6627D_0 - 16000 = 0$$

or $$D_0^3 - 60D_0^2 - 5027D_0 - 16000 = 0 \qquad \ldots(ii)$$

The equation $(ii)$ is solved by trial and error method.

(*i*) Let $D_0 = 100$ mm.

Substituting this value of $D_0$ in the L.H.S. of equation (*ii*), we get

L.H.S. $= 100^3 - 60 \times 100^2 - 5027 \times 100 - 16000$

$= 1000000 - 600000 - 502700 - 16000 = 1000000 - 1118700 = -118700$

(*ii*) Let $D_0 = 110$ mm

Substituting this value in the L.H.S. of equation (*ii*), we get

L.H.S. $= 110^3 - 60 \times 110^2 - 5027 \times 110 - 16000$

$= 1331000 - 726000 - 552970 - 16000 = 1331000 - 1294970 = 36030$

When $D_0 = 100$ mm, the L.H.S. of equation (*ii*), is negative but when $D_0 = 110$ mm, the L.H.S. is positive. Hence the value of $D_0$ lies between 100 and 110 mm. The value of $D_0$ is more nearer to 110 mm as 36030 is less than 118700.

(*iii*) Let $D_0 = 108$ mm.

Substituting this value in the L.H.S. of equation (*ii*), we get

L.H.S. $= 108^3 - 6 \times 108^2 - 5027 \times 108 - 16000$

$= 1259910 - 699840 - 542916 - 16000 = 1259910 - 1258716 = 1194$

The value of $D_0$ will be slightly less than 108 mm, which may be taken as 107.5 mm. **Ans.**

**Problem 16.7.** *A hollow shaft of external diameter 120 mm transmits 300 kW power at 200 r.p.m. Determine the maximum internal diameter if the maximum stress in the shaft is not to exceed 60 N/mm².* (AMIE, Summer 1990)

**Sol.** Given :

External dia., $D_0 = 120$ mm

Power, $P = 300$ kW $= 300{,}000$ W

Speed, $N = 200$ r.p.m.

Max. shear stress, $\tau = 60$ N/mm²

Let $D_i$ = Internal dia. of shaft

Using equation (16.7),

$$P = \frac{2\pi NT}{60} \quad \text{or} \quad 300{,}000 = \frac{2\pi \times 200 \times T}{60}$$

$$\therefore \quad T = \frac{300{,}000 \times 60}{2\pi \times 200} = 14323.9 \text{ Nm}$$

$$= 14323.9 \times 1000 \text{ Nmm} = 14323900 \text{ Nmm}$$

Now using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

$$\text{or} \quad 14323900 = \frac{\pi}{16} \times 60 \times \frac{(120^4 - D_i^4)}{120}$$

$$\text{or} \quad \frac{14323900 \times 16 \times 120}{\pi \times 60} = 120^4 - D_i^4$$

$$145902000 = 207360000 - D_i^4$$

$$\text{or} \quad D_i^4 = 207360000 - 145902000 = 61458000$$

$$\therefore \quad D_i = (61458000)^{1/4} = 88.5 \text{ mm.} \quad \textbf{Ans.}$$

**Problem 16.8.** *Find the maximum shear stress induced in a solid circular shaft of diameter 15 cm when the shaft transmits 150 kW power at 180 r.p.m.*

**Sol.** Given :

Diameter of shaft, $D = 15$ cm $= 150$ mm

Power transmitted, $P = 150$ kW $= 150 \times 10^3$ W

Speed of shaft, $N = 180$ r.p.m.

Let $\tau$ = Maximum shear stress induced in the shaft

Power transmitted is given by equation (16.7) as

$$P = \frac{2\pi NT}{60}$$

$$150 \times 10^3 = \frac{2\pi \times 180 \times T}{60}$$

$$\therefore \quad T = \frac{150 \times 10^3 \times 60}{2\pi \times 180} = 7957.7 \text{ Nm} = 7957700 \text{ Nmm}$$

Now using equation (16.4) as,

$$T = \frac{\pi}{16}\tau D^3$$

$$7957700 = \frac{\pi}{16} \times \tau \times 150^3$$

$$\therefore \quad \tau = \frac{16 \times 7957700}{\pi \times 150^3} = \mathbf{12\ N/mm^2.} \quad \textbf{Ans.}$$

**Problem 16.9.** *A solid cylindrical shaft is to transmit 300 kW power at 100 r.p.m.*

*(a) If the shear stress is not to exceed 80 N/mm², find its diameter.*

*(b) What percent saving in weight would be obtained if this shaft is replaced by a hollow one whose internal diameter equals to 0.6 of the external diameter, the length, the material and maximum shear stress being the same ?* (AMIE, Winter 1983)

**Sol.** Given :

Power, $P = 300$ kW $= 300 \times 10^3$ W

Speed, $N = 100$

Max. shear stress, $\tau = 80$ N/mm²

(*a*) Let $D$ = Dia. of solid shaft

Using equation (16.7),

$$P = \frac{2\pi NT}{60}$$

$$300 \times 10^3 = \frac{2\pi \times 100 \times T}{60}$$

$$\therefore \quad T = \frac{300 \times 10^3 \times 60}{2\pi \times 100} = 28647.8 \text{ Nm} = 28647800 \text{ Nmm}$$

Now using equation (16.4),

$$T = \frac{\pi}{16} \times \tau \times D^3 \quad \text{or} \quad 28647800 = \frac{\pi}{16} \times 80 \times D^3$$

$$\therefore \quad D = \left(\frac{16 \times 28647800}{\pi \times 80}\right)^{1/3} = 121.8 \text{ mm}$$

$$= \text{say } \mathbf{122.0\ mm.} \quad \textbf{Ans.}$$

*(b) Percent saving in weight*

Let $D_0$ = External dia. of hollow shaft

$D_i$ = Internal dia. of hollow shaft

$= 0.6 \times D_0$. (given)

The length, material and maximum shear stress in solid and hollow shafts are given the same. Hence torque transmitted by solid shaft is equal to the torque transmitted by hollow shaft. But the torque transmitted by hollow shaft is given by equation (16.6).

∴ Using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

$$= \frac{\pi}{16} \times 800 \times \frac{[D_0^4 - (0.6\ D_0)^4]}{D_0} \qquad (\because\ D_i = 0.6\ D_0)$$

$$= \pi \times 50 \times \frac{[D_0^4 - (0.6\ D_0)^4]}{D_0}$$

But torque transmitted by solid shaft

= 28647800 Nmm

∴ Equating the two torques, we get

$$28647800 = \pi \times 50 \times \left(\frac{0.8704\ D_0^4}{D_0}\right) = \pi \times 50 \times 0.8704\ D_0^3$$

$$\therefore \qquad D_0 = \left(\frac{28647800}{\pi \times 50 \times 0.8704}\right)^{1/3} = 127.6 \text{ mm} = \text{say } 128 \text{ mm}$$

∴ Internal dia. $D_i = 0.6 \times D_0 = 0.6 \times 128 = 76.8$ mm

Now let $W_s$ = Weight of solid shaft,

and $W_h$ = Weight of hollow shaft.

Then $W_s$ = Weight density × Area of solid shaft × Length

$$= w \times \frac{\pi}{4} D^2 \times L \qquad \text{(where } w = \text{weight density)}$$

Similarly $W_h$ = Weight density × Area of hollow shaft × Length

$$= w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L$$

(∵ Both shafts are of same lengths and of same material)

Now percent saving in weight

$$= \frac{W_s - W_h}{W_s} \times 100$$

$$= \frac{w \times \frac{\pi}{4} D^2 \times L - w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L}{w \times \frac{\pi}{4} D^2 \times L} \times 100$$

$$= \frac{D^2 - (D_0^2 - D_i^2)}{D^2} \times 100 \qquad \left(\text{Cancelling } w \times \frac{\pi}{4} \times L\right)$$

$$= \frac{122^2 - (128^2 - 75.8^2)}{122^2} \times 100 = \frac{14884 - (16364 - 5898)}{14884} \times 100$$

$$= \frac{14884 - 10486}{14884} \times 100 = \mathbf{29.55\%.} \quad \textbf{Ans.}$$

**Problem 16.10.** *A solid steel shaft has to transmit 75 kW at 200 r.p.m. Taking allowable shear stress as 70 N/mm², find suitable diameter for the shaft, if the maximum torque transmitted at each revolution exceeds the mean by 30%.* (AMIE, Summer 1978)

**Sol.** Given :

Power transmitted, $P = 75 \text{ kW} = 75 \times 10^3 \text{ W}$

R.P.M. of the shaft, $N = 200$

Shear stress, $\tau = 70 \text{ N/mm}^2$

Let $T$ = Mean torque transmitted

$T_{max}$ = Maximum torque transmitted = $1.3\,T$

$D$ = Suitable diameter of the shaft

Power is given by the relation,

$$= \frac{2\pi NT}{60}$$

or
$$75 \times 10^3 = \frac{2\pi \times 200 \times T}{60}$$

∴
$$T = \frac{75 \times 10^3 \times 60}{2\pi \times 200} = 3580.98 \text{ Nm} = 3580980 \text{ Nmm}$$

∴
$$T_{max} = 1.3\,T = 1.3 \times 3580980 = 4655274 \text{ Nmm.}$$

Maximum torque transmitted by a solid shaft is given by equation (16.4) as,

$$T_{max} = \frac{\pi}{16} \times \tau \times D^3$$

or
$$4655274 = \frac{\pi}{16} \times 70 \times D^3$$

∴
$$D = \left(\frac{16 \times 4655274}{\pi \times 70}\right)^{1/3} = \mathbf{69.57 \text{ mm} \simeq 70 \text{ mm.}} \quad \textbf{Ans.}$$

## 16.6. EXPRESSION FOR TORQUE IN TERMS OF POLAR MOMENT OF INERTIA

Polar moment of inertia of a plane area is defined as the moment of inertia of the area about an axis perpendicular to the plane of the figure and passing through the C.G. of the area. It is denoted by symbol $J$.

The torque in terms of polar moment of inertia ($J$) is obtained from equation [16.3 ($A$)] of Art. 16.3.

The moment ($dT$) on the circular ring is given by equation [16.3 ($A$)] as

$$dT = \frac{\tau}{R} 2\pi r^3 dr = \frac{\tau}{R} 2\pi r \times r^2 dr = \frac{\tau}{R} r^2 \times 2\pi r \times dr$$

$$= \frac{\tau}{R} r^2 \, dA \qquad (\because \; dA = 2\pi r \, dr \text{ see Fig. 16.3})$$

$\therefore$ Total torque,

$$T = \int_0^R dT = \int_0^R \frac{\tau}{R} r^2 \, dA = \frac{\tau}{R} \int_0^R r^2 \, dA \qquad \ldots(i)$$

But $r^2 \, dA$ = Moment of inertia of the elementary ring about an axis perpendicular to the plane of Fig. 16.3 and passing through the centre of the circle.

$\therefore$ $\int_0^R r^2 \, dA$ = Moment of inertia of the circle about an axis perpendicular to the plane of the circle and passing through the centre of the circle

$= \text{Polar moment of inertia } (J) = \frac{\pi}{32} D^4.$

Hence equation (*i*) becomes as

$$T = \frac{\tau}{R} \times J \qquad \left(\because \; J = \int_0^R r^2 \, dA\right)$$

$$\therefore \qquad \frac{T}{J} = \frac{\tau}{R} \qquad \text{...(16.8)}$$

But from equation (16.1), we have

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

$$\therefore \qquad \frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L} \qquad \text{...(16.9)}$$

where $C$ = Modulus of rigidity

$\theta$ = Angle of twist in radiation

$L$ = Length of the shaft.

### 16.7. POLAR MODULUS

Polar modulus is defined as the ratio of the polar moment of inertia to the radius of the shaft. It is also called torsional section modulus. It is denoted by $Z_p$. Mathematically,

$$Z_p = \frac{J}{R}$$

(*a*) For a solid shaft, $J = \frac{\pi}{32} D^4$

$$\therefore \qquad Z_p = \frac{\frac{\pi}{32} D^4}{R} = \frac{\frac{\pi}{32} D^4}{D/2} = \frac{\pi}{16} D^3 \qquad \text{...(16.10)}$$

(*b*) For a hollow shaft, $J = \frac{\pi}{32} (D_0^4 - D_i^4)$ ...(16.11)

$$\therefore \qquad Z_p = \frac{\frac{\pi}{32}[D_0^4 - D_i^4]}{R}$$

(Here $R$ is the outer radius) $\left(\because \; R = \frac{D_0}{2}\right)$

$$= \frac{\frac{\pi}{32}[D_0^4 - D_i^4]}{D_0/2} = \frac{\pi}{16 D_0} \times [D_0^4 - D_i^4] \qquad \text{...(16.12)}$$

### 16.8. STRENGTH OF A SHAFT AND TORSIONAL RIGIDITY

The strength of a shaft means the *maximum torque or maximum power* the shaft can transmit.

Torsional rigidity or stiffness of the shaft is defined as the product of modulus of rigidity ($C$) and polar moment of inertia of the shaft ($J$). Hence mathematically, the torsional rigidity is given as,

Torsional rigidity = $C \times J$.

Torsional rigidity is also defined as the torque required to produce a twist of one radian per unit length of the shaft.

Let a twisting moment $T$ produces a twist of $\theta$ radians in a shaft of length $L$.
Using equation (16.9), we have

$$\frac{T}{J} = \frac{C\theta}{L} \quad \text{or} \quad C \times J = \frac{T \times L}{\theta}$$

But $C \times J$ = Torsional rigidity

$\therefore$ Torsional rigidity $= \frac{T \times L}{\theta}$

If $L$ = one metre and $\theta$ = one radian

Then torsional rigidity = Torque.

**Problem 16.13.** *Determine the diameter of a solid steel shaft which will transmit 90 kW at 160 r.p.m. Also determine the length of the shaft if the twist must not exceed 1° over the entire length. The maximum shear stress is limited to 60 N/mm². Take the value of modulus of rigidity = 8 × 10⁴ N/mm².*

**Sol.** Given :

Power, $P = 90 \text{ kW} = 90 \times 10^3 \text{ W}$

Speed, $N = 160$ r.p.m.

Angle of twist, $\theta = 1°$ or $\frac{\theta}{180}$ radian $\quad \left(\because 1° = \frac{\pi}{180} \text{ radian}\right)$

Max. shear stress, $\tau = 60 \text{ N/mm}^2$

Modulus of rigidity, $C = 8 \times 10^4 \text{ N/mm}^2$

Let $D$ = Diameter of the shaft and

$L$ = Length of the shaft.

*(i) Diameter of the shaft*

Using equation (16.7),

$$P = \frac{2\pi NT}{60}$$

or

$$90 \times 10^3 = \frac{2\pi \times 160 \times T}{60}$$

$$\therefore \quad T = \frac{90 \times 10^3 \times 60}{2\pi \times 160} = 5371.48 \text{ N-m} = 5371.48 \times 10^3 \text{ N-mm}$$

Now using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or

$$5371.48 \times 10^3 = \frac{\pi}{16} \times 60 \times D^3$$

$$\therefore \quad D^3 = \frac{5371.48 \times 10^3 \times 16}{\pi \times 60} = 455945$$

$$\therefore \quad D = (455945)^{1/3} = \mathbf{76.8 \text{ mm.}} \quad \textbf{Ans.}$$

*(ii) Length of the shaft*

Using equation (16.7),

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

or

$$\frac{60}{\left(\frac{76.8}{2}\right)} = \frac{8 \times 10^4 \times \pi}{L \times 180} \quad \left(\because R = \frac{D}{2} = \frac{76.8}{2} \text{ mm}, \theta = \frac{\pi}{180} \text{ radian}\right)$$

or

$$L = \frac{8 \times 10^4 \times \pi \times 76.8}{60 \times 180 \times 2} = \mathbf{893.6 \text{ mm.}} \quad \textbf{Ans.}$$

# MODULE 4

# STRESSES IN BEAMS

When some external load acts on a beam, the shear force and bending moments are set up at all sections of the beam. Due to the shear force and bending moment, the beam undergoes certain deformation. The material of the beam will offer resistance or stresses against these deformations. These stresses with certain assumptions can be calculated. The stresses introduced by bending moment are known as *bending stresses*. In this chapter, the theory of pure bending, expression for bending stresses, bending stress in symmetrical and unsymmetrical sections, strength of a beam and composite beams will be discussed.

## 7.2. PURE BENDING OR SIMPLE BENDING

If a length of a beam is subjected to a constant bending moment and no shear force (*i.e.*, zero shear force), then the stresses will be set up in that length of the beam due to B.M. only and that length of the beam is said to be in *pure bending* or simple bending. The stresses set up in that length of beam are known as bending stresses.

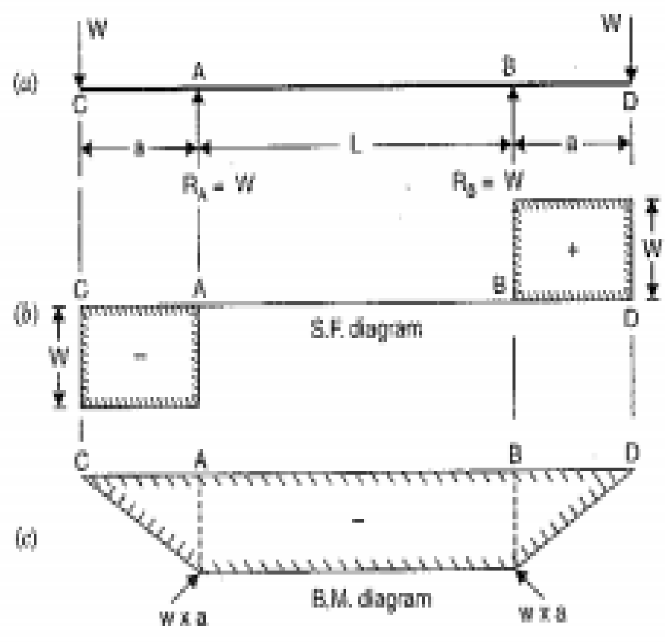



Fig. 7.1

A beam simply supported at *A* and *B* and overhanging by same length at each support is shown in Fig. 7.1. A point load *W* is applied at each end of the overhanging portion. The

S.F. and B.M. for the beam are drawn as shown in Fig. 7.1 (*b*) and Fig. 7.1 (*c*) respectively. From these diagrams, it is clear that there is no shear force between *A* and *B* but the B.M. between *A* and *B* is constant.

This means that between *A* and *B*, the beam is subjected to a constant bending moment only. This condition of the beam between *A* and *B* is known as pure bending or simple bending.

## 7.3. THEORY OF SIMPLE BENDING WITH ASSUMPTIONS MADE

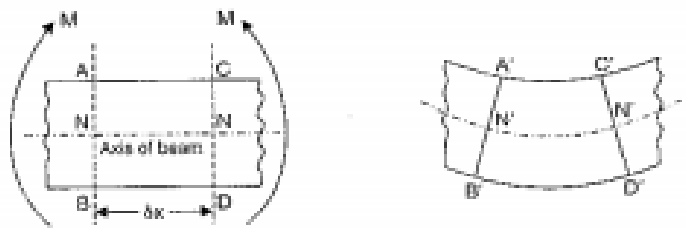
Before discussing the theory of simple bending, let us see the assumptions made in the theory of simple bending. The following are the important assumptions :

1. The material of the beam is homogeneous* and isotropic**.
2. The value of Young's modulus of elasticity is the same in tension and compression.
3. The transverse sections which were plane before bending, remain plane after bending also.
4. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common centre of curvature.
5. The radius of curvature is large compared with the dimensions of the cross-section.
6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.

### Theory of Simple Bending

Fig. 7.2 (*a*) shows a part of a beam subjected to simple bending. Consider a small length $\delta x$ of this part of beam. Consider two sections *AB* and *CD* which are normal to the axis of the beam *N* – *N*. Due to the action of the bending moment, the part of length $\delta x$ will be deformed as shown in Fig. 7.2 (*b*). From this figure, it is clear that all the layers of the beam, which were originally of the same length, do not remain of the same length any more.

The top layer such as *AC* has deformed to the shape *A'C'*. This layer has been shortened in its length. The bottom layer *BD* has deformed to the shape *B'D'*. This layer has been elongated. From the Fig. 7.2 (*b*), it is clear that some of the layers have been shortened while some of them are elongated. At a level between the top and bottom of the beam, there will be a layer which is neither shortened nor elongated. This layer is known as *neutral layer* or *neutral*




(*a*) Before bending (*b*) After bending

Fig. 7.2

*surface*. This layer in Fig. 7.2 (*b*) is shown by $N' - N'$ and in Fig. 7.2 (*a*) by $N - N$. The line of intersection of the neutral layer on a cross-section of a beam is known as *neutral axis* (written as N.A.).

The layers above $N - N$ (or $N' - N'$) have been shortened and those below, have been elongated. Due to the decrease in lengths of the layers above $N - N$, these layers will be subjected to compressive stresses. Due to the increase in the lengths of layers below $N - N$, these layers will be subjected to tensile stresses.

We also see that the top layer has been shortened maximum. As we proceed towards the layer $N - N$, the decrease in length of the layers decreases. At the layer $N - N$, there is no change in length. This means the compressive stress will be maximum at the top layer. Similarly the increase in length will be maximum at the bottom layer. As we proceed from bottom layer towards the layer $N - N$, the increase in length of layers decreases. Hence the amount by which a layer increases or decreases in length, depends upon the position of the layer with respect to $N - N$. This theory of bending is known as theory of simple bending.

### 7.4. EXPRESSION FOR BENDING STRESS

Fig. 7.3 (*a*) shows a small length $\delta x$ of a beam subjected to a simple bending. Due to the action of bending, the part of length $\delta x$ will be deformed as shown in Fig. 7.3 (*b*). Let $A'B'$ and $C'D'$ meet at $O$.

Let $R$ = Radius of neutral layer $N'N'$

$\theta$ = Angle subtended at O by $A'B'$ and $C'D'$ produced.

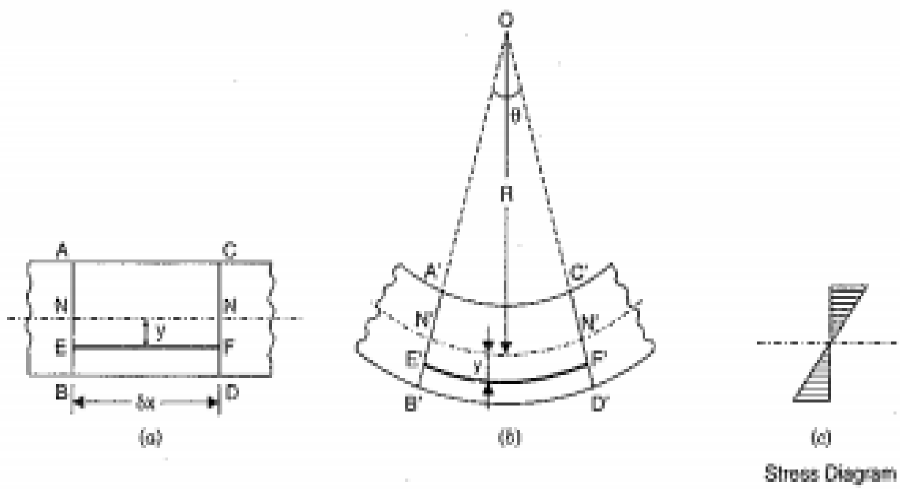



Fig. 7.3

**7.4.1. Strain Variation Along the Depth of Beam.** Consider a layer $EF$ at a distance $y$ below the neutral layer $NN$. After bending this layer will be elongated to $E'F'$.

Original length of layer $EF = \delta x$.

Also length of neutral layer $NN = \delta x$.

After bending, the length of neutral layer $N'N'$ will remain unchanged. But length of layer $E'F'$ will increase. Hence

$$N'N' = NN = \delta x.$$

Now from Fig. 7.3 (*b*),

$$N'N' = R \times \theta$$

and $\quad E'F' = (R + y) \times \theta \quad$ ($\because$ Radius of $E'F' = R + y$)

But $\quad N'N' = NN = \delta x.$

Hence $\quad \delta x = R \times \theta$

$\therefore$ Increase in the length of the layer $EF$

$$= E'F' - EF = (R + y)\,\theta - R \times \theta \quad (\because EF = \delta x = R \times \theta)$$

$$= y \times \theta$$

$\therefore$ Strain in the layer $EF$

$$= \frac{\text{Increase in length}}{\text{Original length}}$$

$$= \frac{y \times \theta}{EF} = \frac{y \times \theta}{R \times \theta} \quad (\because EF = \delta x = R \times \theta)$$

$$= \frac{y}{R}$$

As $R$ is constant, *hence the strain in a layer is proportional to its distance from the neutral axis*. The above equation shows the variation of strain along the depth of the beam. The variation of strain is linear.

**7.4.2. Stress Variation**

Let $\quad \sigma$ = Stress in the layer $EF$

$\quad E$ = Young's modulus of the beam

Then
$$E = \frac{\text{Stress in the layer } EF}{\text{Strain in the layer } EF}$$

$$= \frac{\sigma}{\left(\frac{y}{R}\right)} \quad \left(\because \text{Strain in } EF = \frac{y}{R}\right)$$

$\therefore$
$$\sigma = E \times \frac{y}{R} = \frac{E}{R} \times y \quad \text{...(7.1)}$$

Since $E$ and $R$ are constant, *therefore stress in any layer is directly proportional to the distance of the layer from the neutral layer*. The equation (7.1) shows the variation of stress along the depth of the beam. The variation of stress is linear.

In the above case, all layers below the neutral layer are subjected to tensile stresses whereas the layers above neutral layer are subjected to compressive stresses. The Fig. 7.3 (*c*) shows the stress distribution.

The equation (7.1) can also be written as

$$\frac{\sigma}{y} = \frac{E}{R} \quad \text{...(7.2)}$$

## 7.5. NEUTRAL AXIS AND MOMENT OF RESISTANCE

The neutral axis of any transverse section of a beam is defined as the line of intersection of the neutral layer with the transverse section. It is written as N.A.

In Art. 7.4, we have seen that if a section of a beam is subjected to pure sagging moment, then the stresses will be compressive at any point above the neutral axis and tensile below the neutral axis. There is no stress at the neutral axis. The stress at a distance $y$ from the neutral axis is given by equation (7.1) as

$$\sigma = \frac{E}{R} \times y.$$

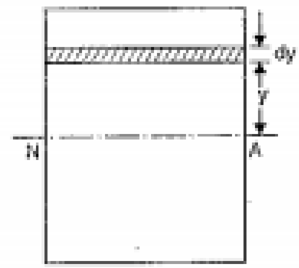


Fig. 7.4

Fig. 7.4 shows the cross-section of a beam. Let N.A. be the neutral axis of the section. Consider a small layer at a distance $y$ from the neutral axis. Let $dA$ = Area of the layer.

Now the force on the layer

= Stress on layer × Area of layer

$= \sigma \times dA$

$$= \frac{E}{R} \times y \times dA \quad \ldots(i) \quad \left(\because \ \sigma = \frac{E}{R} \times y\right)$$

Total force on the beam section is obtained by integrating the above equation.

∴ Total force on the beam section

$$= \int \frac{E}{R} \times y \times dA$$

$$= \frac{E}{R} \int y \times dA \quad (\because \ E \text{ and } R \text{ is constant})$$

But for pure bending, there is no force on the section of the beam (or force is zero).

$$\therefore \quad \frac{E}{R} \int y \times dA = 0$$

or

$$\int y \times dA = 0 \quad \left(\text{as } \frac{E}{R} \text{ cannot be zero}\right)$$

Now $y \times dA$ represents the moment of area $dA$ about neutral axis. Hence $\int y \times dA$ represents the moment of entire area of the section about neutral axis. But we know that moment of any area about an axis passing through its centroid, is also equal to zero. Hence neutral axis coincides with the centroidal axis. Thus the centroidal axis of a section gives the position of neutral axis.

**7.5.1. Moment of Resistance.** Due to pure bending, the layers above the N.A. are subjected to compressive stresses whereas the layers below the N.A. are subjected to tensile stresses. Due to these stresses, the forces will be acting on the layers. These forces will have moment about the N.A. The total moment of these forces about the N.A. for a section is known as moment of resistance of that section.

The force on the layer at a distance $y$ from neutral axis in Fig. 7.4 is given by equation ($i$), as

Force on layer $= \frac{E}{R} \times y \times dA$

Moment of this force about N.A.

= Force on layer × $y$

$$= \frac{E}{R} \times y \times dA \times y$$

$$= \frac{E}{R} \times y^2 \times dA$$

Total moment of the forces on the section of the beam (or moment of resistance)

$$= \int \frac{E}{R} \times y^2 \times dA = \frac{E}{R} \int y^2 \times dA$$

Let $M$ = External moment applied on the beam section. For equilibrium the moment of resistance offered by the section should be equal to the external bending moment.

$$\therefore \quad M = \frac{E}{R} \int y^2 \times dA.$$

But the expression $\int y^2 \times dA$ represents the moment of inertia of the area of the section about the neutral axis. Let this moment of inertia be $I$.

$$\therefore \quad M = \frac{E}{R} \times I \quad \text{or} \quad \frac{M}{I} = \frac{E}{R} \qquad \text{...(7.3)}$$

But from equation (7.2), we have

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \quad \frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \qquad \text{...(7.4)}$$

The equation (7.4) is known as bending equation.

In equation (7.4), the different quantities are expressed in consistent units as given below :

$M$ is expressed in N mm ; $I$ in mm$^4$

$\sigma$ is expressed in N/mm$^2$ ; $y$ in mm

and $E$ is expressed in N/mm$^2$ ; $R$ in mm.

**7.5.2. Condition of Simple Bending.** The equation (7.4) is applicable to a member which is subjected to a constant bending moment and the member is absolutely free from shear force. But in actual practice, a member is subjected to such loading that the B.M. varies from section to section and also the shear force is not zero. But shear force is zero at a section where bending moment is maximum. *Hence the condition of simple bending* may be assumed to be satisfied at such a section. Hence the stresses produced due to maximum bending moment, are obtained from equation (7.4) as the shear forces at these sections are generally zero. Hence the theory and equations discussed in the above articles are quite sufficient and give results which enables the engineers to design beams and structures and calculate their stresses and strains with a reasonable degree of approximation where B.M. is maximum.

## 7.6. BENDING STRESSES IN SYMMETRICAL SECTIONS

The neutral axis (N.A.) of a symmetrical section (such as circular, rectangular or square) lies at a distance of $d/2$ from the outermost layer of the section where $d$ is the diameter (for a circular section) or depth (for a rectangular or a square section). There is no stress at the neutral axis. But the stress at a point is directly proportional to its distance from the neutral axis. The maximum stress takes place at the outermost layer. For a simply supported beam, there is a compressive stress above the neutral axis and a tensile stress below it. If we plot these stresses, we will get a figure as shown in Fig. 7.5.

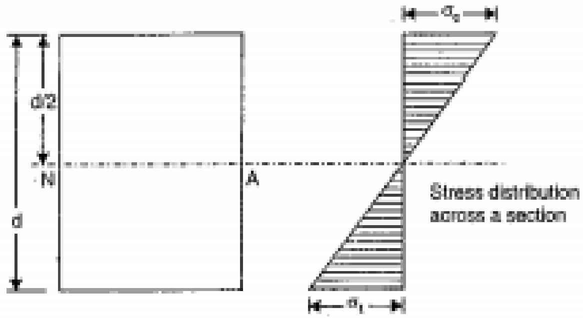



Fig. 7.5

**Problem 7.1.** *A steel plate of width 120 mm and of thickness 20 mm is bent into a circular arc of radius 10 m. Determine the maximum stress induced and the bending moment which will produce the maximum stress. Take $E = 2 \times 10^5$ N/mm².*

**Sol.** Given :

Width of plate, $b = 120$ mm

Thickness of plate, $t = 20$ mm

∴ Moment of inertia, $I = \frac{bt^3}{12} = \frac{120 \times 20^3}{12} = 8 \times 10^4 \text{ mm}^4$

Radius of curvature, $R = 10 \text{ m} = 10 \times 10^3$ mm

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$

Let $\sigma_{max}$ = Maximum stress induced, and

$M$ = Bending moment.

Using equation (7.2), $\frac{\sigma}{y} = \frac{E}{R}$

$$\therefore \quad \sigma = \frac{E}{R} \times y \quad ...(i)$$

Equation (*i*) gives the stress at a distance $y$ from N.A.

Stress will be maximum, when $y$ is maximum. But $y$ will be maximum at the top layer or bottom layer.

$$\therefore \quad y_{max} = \frac{t}{2} = \frac{20}{2} = 10 \text{ mm.}$$

Now equation (*i*) can be written as

$$\sigma_{max} = \frac{E}{R} \times y_{max}$$

$$= \frac{2 \times 10^5}{10 \times 10^3} \times 10 = \mathbf{200 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

From equation (7.4), we have

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad M = \frac{E}{R} \times I = \frac{2 \times 10^5}{10 \times 10^3} \times 8 \times 10^4$$

$$= 16 \times 10^5 \text{ N mm} = \mathbf{1.6 \text{ kNm}}. \quad \textbf{Ans.}$$

**Problem 7.2.** *Calculate the maximum stress* induced in a cast iron pipe of external diameter 40 mm, of internal diameter 20 mm and of length 4 metre when the pipe is supported at its ends and carries a point load of 80 N at its centre.*

**Sol.** Given :

External dia., $D = 40$ mm

Internal dia., $d = 20$ mm

Length, $L = 4$ m $= 4 \times 1000 = 4000$ mm

Point load, $W = 80$ N

In case of simply supported beam carrying a point load at the centre, the maximum bending moment is at the centre of the beam.

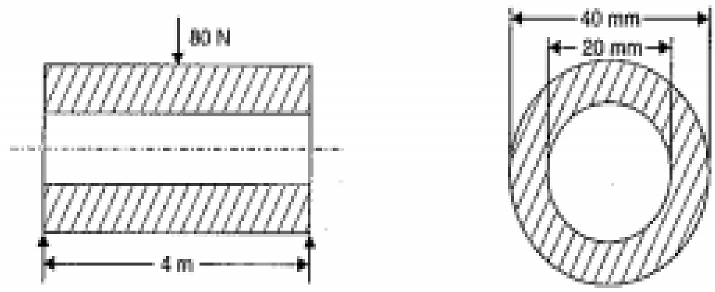



Fig. 7.6

And maximum B.M. $= \dfrac{W \times L}{4}$

$\therefore$ Maximum B.M. $= \dfrac{80 \times 4000}{4} = 8 \times 10^4$ Nmm

$\therefore$ $M = 8 \times 10^4$ Nmm

Fig. 7.6 (*b*) shows the cross-section of the pipe.

Moment of inertia of hollow pipe,

$$I = \frac{\pi}{64}[D^4 - d^4]$$

$$= \frac{\pi}{64}[40^4 - 20^4] = \frac{\pi}{64}[2560000 - 160000]$$

$$= 117809.7 \text{ mm}^4$$

Now using equation (7.4),

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{...(i)}$$

when $y$ is maximum, stress will be maximum. But $y$ is maximum at the top layer from the N.A.

$$\therefore \quad y_{max} = \frac{D}{2} = \frac{40}{2} = 20 \text{ mm}$$

The above equation (*i*) can be written as

$$\frac{M}{I} = \frac{\sigma_{max}}{y_{max}}$$

$$\therefore \quad \sigma_{max} = \frac{M}{I} \times y_{max}$$

$$= \frac{8 \times 10^4 \times 20}{117809.7} = \mathbf{13.58\ N/mm^2.} \quad \textbf{Ans.}$$

## 7.7. SECTION MODULUS

Section modulus is defined as the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis. It is denoted by the symbol $Z$. Hence mathematically section modulus is given by,

$$Z = \frac{I}{y_{max}} \qquad \text{...(7.5)}$$

where $I$ = M.O.I. about neutral axis

and $y_{max}$ = Distance of the outermost layer from the neutral axis.

From equation (7.4), we have

$$\frac{M}{I} = \frac{\sigma}{y}$$

The stress $\sigma$ will be maximum, when $y$ is maximum. Hence above equation can be written as

$$\frac{M}{I} = \frac{\sigma_{max}}{y_{max}}$$

$$\therefore \quad M = \sigma_{max} \cdot \frac{I}{y_{max}}$$

But $\dfrac{I}{y_{max}} = Z$

$$\therefore \quad M = \sigma_{max} \cdot Z \qquad \text{...(7.6)}$$

In the above equation, $M$ is the maximum bending moment (or moment of resistance offered by the section). Hence moment of resistance offered the section is maximum when section modulus $Z$ is maximum. Hence section modulus represent the strength of the section.

## 7.8. SECTION MODULUS FOR VARIOUS SHAPES OR BEAM SECTIONS

1. *Rectangular Section*

Moment of inertia of a rectangular section about an axis through its C.G. (or through N.A.) is given by,

$$I = \frac{bd^3}{12}$$

Distance of outermost layer from N.A. is given by,

$$y_{max} = \frac{d}{2}$$

$\therefore$ Section modulus is given by,

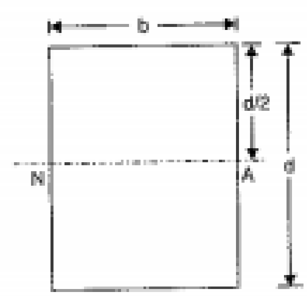



Fig. 7.7

$$Z = \frac{I}{y_{max}} = \frac{bd^3}{12 \times \left(\frac{d}{2}\right)} = \frac{bd^3}{12} \times \frac{2}{d} = \frac{bd^2}{6} \qquad ...(7.7)$$

*2. Hollow Rectangular Section*

Here $$I = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$= \frac{1}{12}[BD^3 - bd^3]$$

$$y_{max} = \frac{D}{2}$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\frac{1}{12}[BD^3 - bd^3]}{\left(\frac{D}{2}\right)}$$

$$= \frac{1}{6D}[BD^3 - bd^3] \qquad ...(7.8)$$

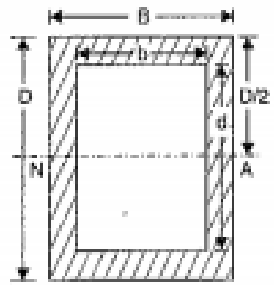


Fig. 7.8

*3. Circular Section*

For a circular section,

$$I = \frac{\pi}{64} d^4 \quad \text{and} \quad y_{max} = \frac{d}{2}$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} d^4}{\left(\frac{d}{2}\right)} = \frac{\pi}{32} d^3 \qquad ...(7.9)$$

*4. Hollow Circular Section*

Here $$I = \frac{\pi}{64}[D^4 - d^4]$$

and $$y_{max} = \frac{D}{2}$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64}[D^4 - d^4]}{\left(\frac{D}{2}\right)}$$

$$= \frac{\pi}{32D}[D^4 - d^4] \qquad ...(7.10)$$

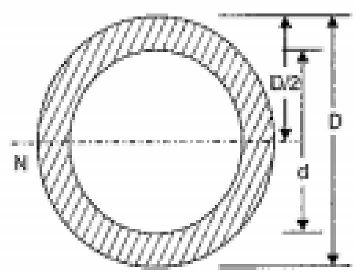


Fig. 7.9

**Problem 7.3.** *A cantilever of length 2 metre fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm × 60 mm, find the stress at the failure.*

**Sol.** Given :

Length, $L = 2 \text{ m} = 2 \times 10^3 \text{ mm}$

Load, $W = 2 \text{ kN} = 2000 \text{ N}$

Section of beam is 40 mm × 60 mm.

∴ Width of beam, $b = 40$ mm

Depth of beam, $d = 60$ mm

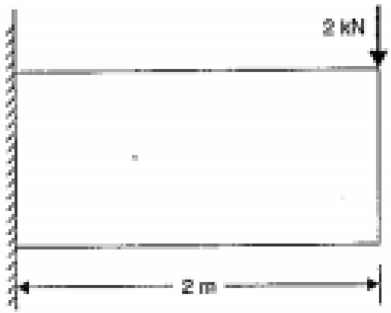



Fig. 7.10

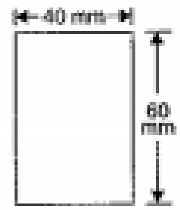



Fig. 7.10 (a)

Fig. 7.10 (a) shows the section of the beam.

Section modulus of a rectangular section is given by equation (7.7).

$$\therefore \quad Z = \frac{bd^2}{6} = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3$$

Maximum bending moment for a cantilever shown in Fig. 7.10 is at the fixed end.

$$\therefore \quad M = W \times L = 2000 \times 2 \times 10^3 = 4 \times 10^6 \text{ Nmm}$$

Let $\sigma_{max}$ = Stress at the failure

Using equation (7.6), we get

$$M = \sigma_{max} \cdot Z$$

$$\therefore \quad \sigma_{max} = \frac{M}{Z} = \frac{4 \times 10^6}{24000} = \mathbf{166.67 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

**Problem 7.4.** *A rectangular beam 200 mm deep and 300 mm wide is simply supported over a span of 8 m. What uniformly distributed load per metre the beam may carry, if the bending stress is not to exceed 120 N/mm².*

**Sol.** Given :

Depth of beam, $d = 200$ mm

Width of beam, $b = 300$ mm

Length of beam, $L = 8$ m

Max. bending stress, $\sigma_{max} = 120$ N/mm²

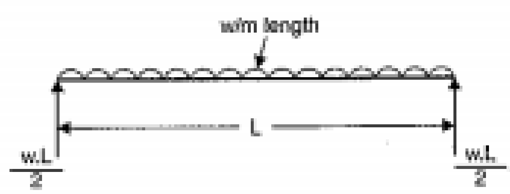



Fig. 7.11

Let $w$ = Uniformly distributed load per m length over the beam

(Fig. 7.11 (a) shows the section of the beam).

Section modulus for a rectangular section is given by equation (7.7).

$$\therefore \quad Z = \frac{bd^2}{6} = \frac{300 \times 200^2}{6} = 2000000 \text{ mm}^3$$

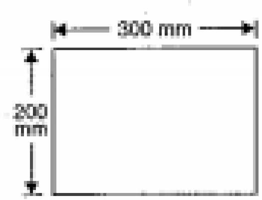



Fig. 7.11 (a)

Max. B.M. for a simply supported beam carrying uniformly distributed load as shown in Fig. 7.11 is at the centre of the beam. It is given by

$$M = \frac{w \times L^2}{8} = \frac{w \times 8^2}{8} \qquad (\because \; L = 8 \text{ m})$$

$$= 8w \text{ Nm} = 8w \times 1000 \text{ Nmm}$$

$$= 8000w \text{ Nmm} \qquad (\because \; 1 \text{ m} = 1000 \text{ mm})$$

Now using equation (7.6), we get

$$M = \sigma_{max} . Z$$

or $$8000w = 120 \times 2000000$$

$\therefore$ $$w = \frac{120 \times 2000000}{8000} = 30 \times 1000 \text{ N/m} = \textbf{30 kN/m.} \quad \textbf{Ans.}$$

**Problem 7.5.** *A rectangular beam 300 mm deep is simply supported over a span of 4 metres. Determine the uniformly distributed load per metre which the beam may carry, if the bending stress should not exceed 120 N/mm². Take $I = 8 \times 10^6$ mm⁴.*

(Annamalai University, 1991)

**Sol.** Given :

Depth, $d = 300$ mm

Span, $L = 4$ m

Max. bending stress, $\sigma_{max} = 120$ N/mm²

Moment of inertia, $I = 8 \times 10^6$ mm⁴

Let, $w$ = U.D.L. per metre length over the beam in N/m.

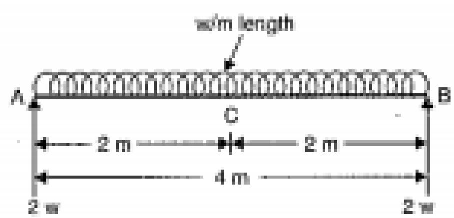



Fig. 7.11 (*b*)

The bending stress will be maximum, where bending moment is maximum. For a simply supported beam carrying U.D.L., the bending moment is maximum at the centre of the beam [*i.e.*, at point *C* of Fig. 7.11 (*b*)]

$\therefore$ $$\text{Max. B.M.} = 2w \times 2 - 2w \times 1$$

$$= 4w - 2w$$

$$= 2w \text{ Nm} \qquad \left(\text{Also } M = \frac{w \times L^2}{8} = \frac{w \times 4^2}{8} = \frac{16w}{8} = 2w\right)$$

$$= 2w \times 1000 \text{ Nmm}$$

or $$M = 2000w \text{ Nmm}$$

Now using equation (7.6), we get

$$M = \sigma_{max} \times Z \qquad ...(i)$$

where $$Z = \frac{I}{y_{max}} = \frac{8 \times 10^6}{150} \qquad \left(\because \; y_{max} = \frac{d}{2} = \frac{300}{2} = 150 \text{ mm}\right)$$

Hence above equation (*i*) becomes as

$$2000w = 120 \times \frac{8 \times 10^6}{150}$$

or $$w = \frac{120 \times 8 \times 10^6}{2000 \times 150} = \textbf{3200 N/m.} \quad \textbf{Ans.}$$

**Problem 7.6.** *A square beam 20 mm × 20 mm in section and 2 m long is supported at the ends. The beam fails when a point load of 400 N is applied at the centre of the beam. What uniformly distributed load per metre length will break a cantilever of the same material 40 mm wide, 60 mm deep and 3 m long ?*

**Sol.** Given :

Depth of beam, $d = 20$ mm

Width of beam, $b = 20$ mm

Length of beam, $L = 2$ m

Point load, $W = 400$ N

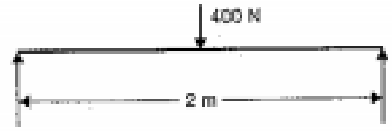



Fig. 7.12

In this problem, the maximum stress for the simply supported beam is to be calculated first. As the material of the cantilever is same as that of simply supported beam, hence maximum stress for the cantilever will also be same as that of simply supported beam.

Fig. 7.12 (*a*) shows the section of beam.

The section modulus for the rectangular section of simply supported beam is given by equation (7.7).

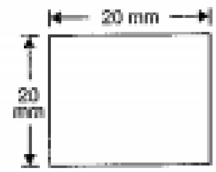



Fig. 7.12 (*a*)

$$\therefore \qquad Z = \frac{bd^2}{6} = \frac{20 \times 20^2}{6} = \frac{4000}{3} \text{ mm}^3$$

Max. B.M. for a simply supported beam carrying a point load at the centre (as shown in Fig. 7.12) is given by,

$$M = \frac{w \times L}{4} = \frac{400 \times 2}{4} = 200 \text{ Nm}$$

$$= 200 \times 1000 = 200000 \text{ Nmm}$$

Let $\sigma_{max}$ = Max. stress induced

Now using equation (7.6), we get

$$M = \sigma_{max} \cdot Z$$

or
$$200000 = \sigma_{max} \times \frac{4000}{3}$$

$$\therefore \qquad \sigma_{max} = \frac{200000 \times 3}{4000} = 150 \text{ N/mm}^2$$

Now let us consider the cantilever as shown in Fig. 7.13.

Let $w$ = Uniformly distributed load per m run.

Maximum stress will be same as in case of simply supported beam.

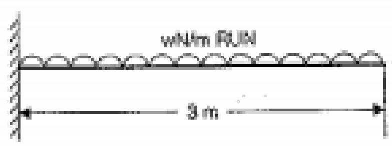



Fig. 7.13

$$\therefore \qquad \sigma_{max} = 150 \text{ N/mm}^2$$

Width of cantilever, $b = 40$ mm

Depth of cantilever, $d = 60$ mm

Length of cantilever, $L = 3$ m

Fig. 7.13 (*a*) shows the section of cantilever beam.

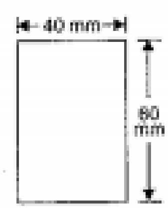



Fig. 7.13 (*a*)

Section modulus of rectangular section of cantilever $= \dfrac{bd^2}{6}$

$$\therefore \qquad Z = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3$$

Maximum B.M. for a cantilever

$$= \frac{wL^2}{2} = \frac{w \times 3^2}{2} = 4.5w \text{ Nm} = 4.5 \times 1000w \text{ Nmm}$$

$\therefore \quad M = 4.5 \times 1000w$ Nmm

Now using equation (7.6), we get

$$M = \sigma_{max} \,.\, Z$$

or $\quad 4.5 \times 1000w = 150 \times 24000$

$\therefore \quad w = \frac{150 \times 24000}{4.5 \times 1000} = \mathbf{800\ N/m.}$ **Ans.**

**Problem 7.7.** *A beam is simply supported and carries a uniformly distributed load of 40 kN/m run over the whole span. The section of the beam is rectangular having depth as 500 mm. If the maximum stress in the material of the beam is 120 N/mm² and moment of inertia of the section is 7 × 10⁸ mm⁴, find the span of the beam.*

**Sol.** Given :

U.D.L., $\quad w = 40$ kN/m $= 40 \times 1000$ N/m

Depth, $\quad d = 500$ mm

Max. stress, $\quad \sigma_{max} = 120$ N/mm²

M.O.I. of section, $\quad I = 7 \times 10^8$ mm⁴

Let $\quad L$ = Span of simply supported beam.

Section modulus of the section is given by equation (7.5), as

$$Z = \frac{I}{y_{max}}$$

where $y_{max} = \frac{d}{2} = \frac{500}{2} = 250$ mm

$\therefore \quad Z = \frac{7 \times 10^8}{250} = 28 \times 10^5$ mm³

The maximum B.M. for a simply supported beam, carrying a U.D.L. over the whole span is at the centre of the beam and is equal to $\frac{w \,.\, L^2}{8}$.

$\therefore \quad M = \frac{w \,.\, L^2}{8} = \frac{40000 \times L^2}{8}$

$\quad = 5000L^2$ Nm $= 5000L^2 \times 1000$ Nmm

Now using equation (7.6), we get

$$M = \sigma_{max} \,.\, Z$$

or $\quad 5000 \times 1000 \times L^2 = 120 \times 28 \times 10^5$

or $\quad L^2 = \frac{120 \times 28 \times 10^5}{5000 \times 1000} = 2.4 \times 28$

$\therefore \quad L = \sqrt{2.4 \times 28} = 8.197$ m say **8.20 m. Ans.**

**Problem 7.8.** *A timber beam of rectangular section is to support a load of 20 kN uniformly distributed over a span of 3.6 m when beam is simply supported. If the depth of section is to be twice the breadth, and the stress in the timber is not to exceed 7 N/mm², find the dimensions of the cross-section.*

*How would you modify the cross-section of the beam, if it carries a concentrated load of 20 kN placed at the centre with the same ratio of breadth to depth ?*

**Sol.** Given :

Total load, $W = 20 \text{ kN} = 20 \times 1000 \text{ N}$

Span, $L = 3.6 \text{ m}$

Max. stress, $\sigma_{max} = 7 \text{ N/mm}^2$

Let $b$ = Breadth of beam in mm

Then depth, $d = 2b$ mm

Section modulus of rectangular beam $= \frac{bd^2}{6}$

$$\therefore \quad L = \frac{b \times (2b)^2}{6} = \frac{2b^3}{6} \text{ mm}^3$$

Maximum B.M., when the simply supported beam carries a U.D.L. over the entire span, is at the centre of the beam and is equal to $\frac{wL^2}{8}$ or $\frac{WL}{8}$.

$$\therefore \quad M = \frac{WL}{8} = \frac{20000 \times 3.6}{8} = 9000 \text{ Nm}$$

$$= 9000 \times 1000 \text{ Nmm}$$

Now using equation (7.6), we get

$$M = \sigma_{max} \cdot Z$$

or
$$9000 \times 1000 = 7 \times \frac{2b^3}{3}$$

or
$$b^3 = \frac{3 \times 9000 \times 1000}{7 \times 2} = 1.92857 \times 10^6$$

$$\therefore \quad b = (1.92857 \times 10^6)^{1/3}$$

$= 124.47$ mm say **124.5 mm. Ans.**

and $d = 2b = 2 \times 124.5 =$ **249 mm. Ans.**

*Dimension of the section when the beam carries a point load at the centre.*

B.M. is maximum at the centre and it is equal to $\frac{W \times L}{4}$ when the beam carries a point load at the centre.

$$\therefore \quad M = \frac{W \times L}{4} = \frac{20000 \times 3.6}{4} = 18000 \text{ Nm}$$

$$= 18000 \times 1000 \text{ Nmm}$$

$$\sigma_{max} = 7 \text{ N/mm}^2$$

and
$$Z = \frac{2b^3}{3} \qquad (\because \text{ In this case also } d = 2b)$$

Using equation (7.6), we get

$$M = \sigma_{max} . Z$$

or
$$18000 \times 1000 = 7 \times \frac{2b^3}{3}$$

$$\therefore \quad b^3 = \frac{3 \times 18000 \times 1000}{7 \times 2} = 3.85714 \times 10^6$$

$\therefore \quad b = (3.85714 \times 10^6)^{1/3} =$ **156.82 mm. Ans.**

and $d = 2 \times 156.82 =$ **313.64 mm. Ans.**

# Shear Stresses in Beams

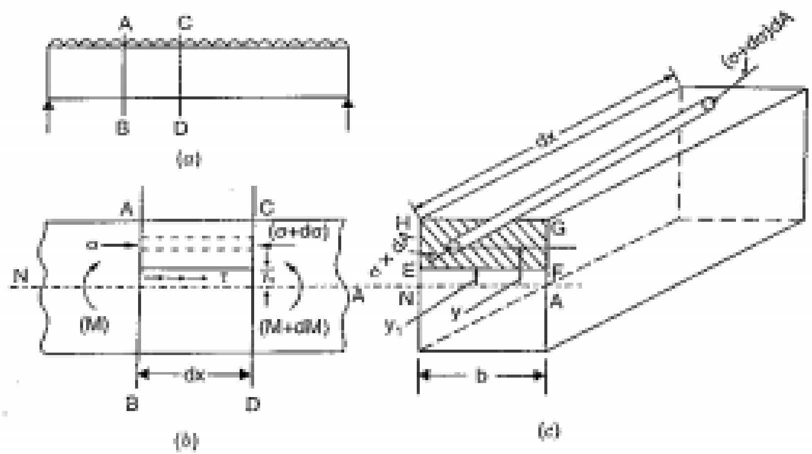
## 8.1. INTRODUCTION

In the last chapter, we have seen that when a part of a beam is subjected to a constant bending moment and zero shear force, then there will be only bending stresses in the beam. The shear stress will be zero as shear stress is equal to shear force divided by the area. As shear force is zero, the shear stress will also be zero.

But in actual practice, a beam is subjected to a bending moment which varies from section to section. Also the shear force acting on the beam is not zero. It also varies from section to section. Due to these shear forces, the beam will be subjected to shear stresses. These shear stresses will be acting across transverse sections of the beam. These transverse shear stresses will produce a complimentary horizontal shear stresses, which will be acting on longitudinal layers of the beam. Hence beam will also be subjected to shear stresses. In this chapter, the distribution of the shear stress across the various sections (such as Rectangular section, Circular section, I-section, T-sections etc.) will be determined.

## 8.2. SHEAR STRESS AT A SECTION

Fig 8.1 (*a*) shows a simply supported beam carrying a uniformly distributed load. For a uniformly distributed load, the shear force and bending moment will vary along the length of the beam. Consider two sections *AB* and *CD* of this beam at a distance *dx* apart.




Area, *A* = Area of *EFGH*

Fig. 8.1

Let at the section $AB$,

$F$ = Shear force

$M$ = Bending moment

and at section $CD$, $F + dF$ = Shear force

$M + dM$ = Bending moment

$I$ = Moment of inertia of the section about the neutral axis.

Let it is required to find the shear stress on the section $AB$ at a distance $y_1$ from the neutral axis. Fig. 8.1 ($c$) shows the cross-section of the beam. On the cross-section of the beam, let $EF$ be a line at a distance $y_1$ from the neutral axis. Now consider the part of the beam above the level $EF$ and between the sections $AB$ and $CD$. This part of the beam may be taken to consists of an infinite number of elemental cylinders each of area $dA$ and length $dx$. Consider one such elemental cylinder at a distance $y$ from the neutral axis.

$\therefore$ $dA$ = Area of elemental cylinder

$dx$ = Length of elemental cylinder

$y$ = Distance of elemental cylinder from neutral axis

Let $\sigma$ = Intensity of bending stress* on the end of the elemental cylinder on the section $AB$

$\sigma + d\sigma$ = Intensity of bending stress on the end of the elemental cylinder on the section $CD$.

The bending stress at distance $y$ from the neutral axis is given by equation (7.6) as

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\therefore \quad \sigma = \frac{M}{I} \times y$$

For a given beam, the bending stress is a function of bending moment and the distance $y$ from neutral axis. Let us find the bending stress on the end of the elemental cylinder at the section $AB$ and also at the section $CD$.

$\therefore$ Bending stress on the end of elemental cylinder on the section $AB$, (where bending moment is $M$) will be

$$\sigma = \frac{M}{I} \times y$$

Similarly, bending stress on the end of elemental cylinder on the section $CD$, (where bending moment is $M + dM$) will be

$$\sigma + d\sigma = \frac{(M + dM)}{I} \times y$$

($\because$ On section $CD$, B.M. = $M + dM$ and bending stress = $\sigma + d\sigma$)

Now let us find the forces on the two ends of the elemental cylinder.

Force on the end of the elemental cylinder on the section $AB$

= Stress × Area of elemental cylinder

= $\sigma \times dA$

$$= \frac{M}{I} \times y \times dA \qquad \left(\because \sigma = \frac{M}{I} \times y\right)$$

Similarly, force on the end of the elemental cylinder on the section $CD$

$$= (\sigma + d\sigma)\, dA$$

$$= \frac{(M + dM)}{I} \times y \times dA \qquad \left[\because \sigma + d\sigma = \frac{(M + dM)}{I} \times y\right]$$

At the two ends of the elemental cylinder, the forces are different. They are acting along the same line but are in opposite direction. Hence there will be unbalanced force on the elemental cylinder.

∴ Net unbalanced force on the elemental cylinder

$$= \frac{(M + dM)}{I} \times y \times dA - \frac{M}{I} \times y \times dA$$

$$= \frac{dM}{I} \times y \times dA \qquad \ldots(i)$$

The total unbalanced force above the level $EF$ and between the two sections $AB$ and $CD$ may be found out by considering all the elemental cylinders between the sections $AB$ and $CD$ and above the level $EF$ (*i.e.*, by integrating the above equation (*i*)).

∴ Total unbalanced force

$$= \int \frac{dM}{I} \times y \times dA = \frac{dM}{I} \int y \times dA$$

$$= \frac{dM}{I} \times A \times \bar{y} \qquad \left(\because \int y \times dA = A \times \bar{y}\right)$$

where $A$ = Area of the section above the level $EF$ (or above $y_1$)

= Area of $EFGH$ as shown in Fig. 8.1 ($c$)

$\bar{y}$ = Distance of the C.G. of the area $A$ from the neutral axis.

Due to the total unbalanced force acting on the part of the beam above the level $EF$ and between the sections $AB$ and $CD$ as shown in Fig. 8.2 ($a$), the beam may fail due to shear. Hence in order the above part may not fail by shear, the horizontal section of the beam at the level $EF$ must offer a shear resistance. This shear resistance at least must be equal to total unbalanced force to avoid failure due to shear.

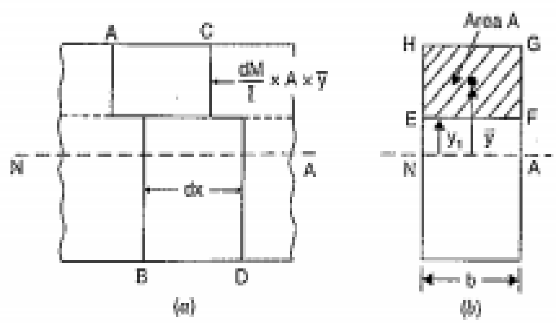



Fig. 8.2

∴ Shear resistance (or shear force) at the level *EF*

$$= \text{Total unbalanced force}$$

$$= \frac{dM}{I} \times A \times \bar{y} \qquad \text{...(ii)}$$

Let $\tau$ = Intensity of horizontal shear at the level *EF*

$b$ = Width of beam at the level *EF*

∴ Area on which $\tau$ is acting

$$= b \times dx$$

∴ Shear force due to $\tau$

$$= \text{Shear stress} \times \text{Shear area}$$

$$= \tau \times b \times dx \qquad \text{...(iii)}$$

Equating the two values of shear force given by equation (*ii*) and (*iii*), we get

$$\tau \times b \times dx = \frac{dM}{I} \times A \times \bar{y}$$

$$\therefore \quad \tau = \frac{dM}{I} \times \frac{A \times \bar{y}}{b \times dx}$$

$$= \frac{dM}{dx} \cdot \frac{A\bar{y}}{I \times b}$$

$$= F \times \frac{A\bar{y}}{I \times b} \qquad \left(\because \frac{dM}{dx} = F = \text{Shear force}\right) \qquad \text{...(8.1)}$$

The shear stress given by equation (8.1) is the horizontal shear stress at the distance $y_1$ from the neutral axis. But by the *principal of complementary shear*, the horizontal shear stress is accompanied by a vertical shear stress $\tau$ of the same quantity.

Sometimes $A \times \bar{y}$ is also expressed as the moment of area $A$ about the neutral axis.

**Note.** In equation (8.1), $b$ is the actual width at the level *EF* (Though here $b$ is same at all levels, in many cases $b$ may not be same at all levels) and $I$ is the total moment of inertia of the section about N.A.

**Problem 8.1.** *A wooden beam 100 mm wide and 150 mm deep is simply supported over a span of 4 metres. If shear force at a section of the beam is 4500 N, find the shear stress at a distance of 25 mm above the N.A.*

**Sol.** Given :

Width, $b = 100$ mm

Depth, $d = 150$ mm

Shear force, $F = 4500$ N

Let $\tau$ = Shear stress at a distance of 25 mm above the neutral axis.

Using equation (8.1), we get

$$\tau = F \cdot \frac{A\bar{y}}{I \cdot b} \qquad \text{...(i)}$$

where $A$ = Area of the beam above $y_1$

$= 100 \times 50 = 5000 \text{ mm}^2$

(Shaded area of Fig. 8.2)

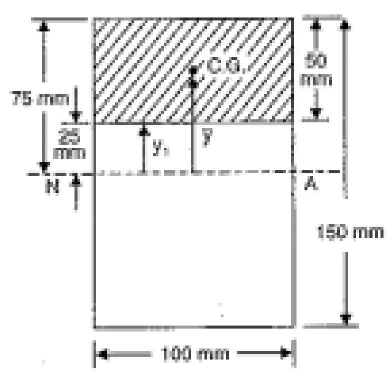



Fig. 8.3

$\bar{y}$ = Distance of the C.G. of the area $A$ from neutral axis

$= 25 + \frac{50}{2} = 50$ mm

$I$ = M.O.I. of the total section

$= \frac{bd^3}{12}$

$= \frac{100 \times 150^3}{12} = 28125000 \text{ mm}^4$

$b$ = Actual width of section at a distance $y_1$ from N.A. = 100 m

Substituting these values in the above equation (*i*), we get

$$\tau = \frac{4500 \times 5000 \times 50}{28125000 \times 100} = \mathbf{0.4 \ N/mm^2}. \quad \textbf{Ans.}$$

**Problem 8.2.** *A beam of cross-section of an isosceles triangle is subjected to a shear force of 30 kN at a section where base width = 150 mm and height = 450 mm. Determine :*

*(i) horizontal shear stress at the neutral axis,*

*(ii) the distance from the top of the beam where shear stress is maximum, and*

*(iii) value of maximum shear stress.*

**Sol.** Given :

Shear force at the section, $F$ = 30 kN = 30,000 N

Base width, $CD$ = 150 mm

Height, $h$ = 450 mm.

(*i*) *Horizontal shear stress at the neutral axis*

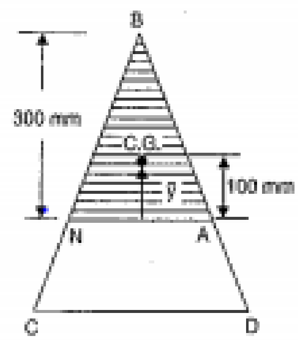



Fig. 8.3 (*a*)

The neutral axis of the triangle is at a distance of $\frac{h}{3}$ from base or $\frac{2h}{3}$ from the apex $B$. Hence distance of neutral axis from $B$ will be $\frac{2 \times 450}{3} = 300$ mm as shown in Fig. 8.3 (*a*). The width of the section at neutral axis is obtained from similar triangles $BCD$ and $BNA$ as

$$\frac{NA}{CD} = \frac{300}{450}$$

or $$NA = \frac{300}{450} \times CD = \frac{300}{450} \times 150 = 100 \text{ mm.}$$

The shear stress at any section is given by equation (8.1) as

$$\tau = F \times \frac{A \times \bar{y}}{I \times b} \quad ...(i)$$

where $\tau$ = Shear stress at the section

$F$ = Shear force = 30,000 N

$A$ = Area above the axis at which shear stress is to be obtained [*i.e.*, shaded area of Fig. 8.3 (*a*)]

$$= \frac{NA \times 300}{2} = \frac{100 \times 300}{2} = 15000 \text{ mm}^2$$

$\bar{y}$ = Distance of the C.G. of the area $A$ from neutral axis

$$= \frac{1}{3} \times 300 = 100 \text{ mm}$$

$I$ = M.O.I. of the total section about neutral axis

$$= \frac{\text{Base width} \times \text{Height}^3}{36} \qquad \left(\because \frac{B \times h^3}{36} \text{ where } B = \text{Base Width of Triangle}\right)$$

$$= \frac{150 \times 450^3}{36} \text{ mm}^4$$

$b$ = Actual width of the section at which shear stress is to be obtained

$= NA = 100$ mm.

Substituting these values in equation (*i*), we get

$$\tau = 30{,}000 \times \frac{15000 \times 100}{\left(\frac{150 \times 450}{36}\right)^3 \times 100} \text{ N/mm}^2$$

$= \mathbf{1.185\ N/mm^2}$. **Ans.**

(*ii*) *The distance from the top of the beam where shear stress is maximum*

Let the shear stress is maximum at the section $EF$ at a distance $x$ from the top of the beam as shown in Fig. 8.3 (*b*). The distance $EF$ is obtained from similar triangles $BEF$ and $BCD$ as

$$\frac{EF}{CD} = \frac{x}{450}$$

$$\therefore \qquad EF = \frac{x}{450} \times CD = \frac{x}{450} \times 150 = \frac{x}{3}.$$

The shear stress at the section $EF$ is given by equation (8.1) as

$$\tau = F \times \frac{A \times \bar{y}}{I \times b} \qquad \ldots(ii)$$

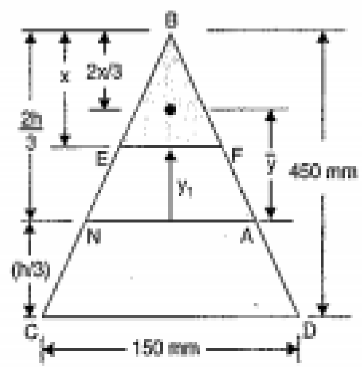


Fig. 8.3 (*b*)

where $F = 30{,}000$ N

$A$ = Area of section above $EF$ *i.e.*, Area of shaded triangle $BEF$

$$= \frac{EF \times x}{2} = \frac{x}{3} \times \frac{x}{2} \qquad \left(\because EF = \frac{x}{3}\right)$$

$$= \frac{x^2}{6}$$

$\bar{y}$ = Distance of C.G. of the Area $A$ from neutral axis

$$= \frac{2h}{3} - \frac{2x}{3} = \frac{2 \times 450}{3} - \frac{2x}{3} = \left(300 - \frac{2x}{3}\right)$$

$I$ = M.O.I. of $\Delta BCD$ about neutral axis

$$= \frac{150 \times 450^3}{36} \text{ mm}^4$$

$$b = \text{Width of section } EF = \frac{x}{3}.$$

Substituting these values in equation (*ii*), we get

$$\tau = \frac{30{,}000 \times \left(\frac{x^2}{6}\right) \times \left(300 - \frac{2x}{3}\right)}{\left(\frac{150 \times 450^3}{36}\right) \times \frac{x}{3}} = 0.0000395x \left(300 - \frac{2x}{3}\right)$$

$$= 0.0000395 \left(300x - \frac{2x^2}{3}\right) \qquad \text{...(iii)}$$

For maximum shear stress $\frac{d\tau}{dx} = 0$

or $$300 - \frac{2}{3} \times 2x = 0 \quad \text{or} \quad 300 = \frac{4x}{3}$$

or $$x = \frac{300 \times 3}{4} = \textbf{225 mm.} \quad \textbf{Ans.}$$

Hence, shear stress is maximum at a distance of 225 mm from the top of the beam.

(*iii*) *Value of Maximum Shear Stress*

The value of maximum shear stress will be obtained by substituting $x = 225$ mm in equation (*iii*).

$\therefore$ Maximum shear stress $= 0.0000395 \left(300 \times 225 - \frac{2}{3} \times 225^2\right)$

$= \textbf{1.333 N/mm}^2$. **Ans.**

# MODULE 5

# SHEAR FORCE & BENDING MOMENT

## 6.1. INTRODUCTION

The algebraic sum of the vertical forces at any section of a beam to the right or left of the section is known as shear force. It is briefly written as S.F. The algebraic sum of the moments of all the forces acting to the right or left of the section is known as bending moment. It is written as B.M. In this chapter, the shear force and bending moment diagrams for different types of beams (*i.e.*, cantilevers, simply supported, fixed, overhanging etc.) for different types of loads (*i.e.*, point load, uniformly distributed loads, varying loads etc.) acing on the beams, will be considered.

## 6.2. SHEAR FORCE AND BENDING MOMENT DIAGRAMS

A shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which shows the variation of the bending moment along the length of the beam.

Before drawing the shear force and bending moment diagrams, we must know the different types of beams and different types of load acting on the beams.

## 6.3. TYPES OF BEAMS

The following are the important types of beams :

1. Cantilever beam,
2. Simply supported beam,
3. Overhanging beam,
4. Fixed beams, and
5. Continuous beam.

**6.3.1. Cantilever Beam.** A beam which is fixed at one end and free at the other end, is known as cantilever beam. Such beam is shown in Fig. 6.1.

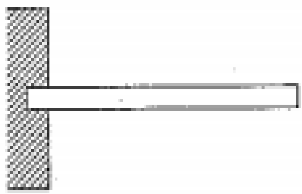


Fig. 6.1

Fig. 6.2

**6.3.2. Simply Supported Beam.** A beam supported or resting freely on the supports at its both ends, is known as simply supported beam. Such beam is shown in Fig. 6.2.

**6.3.3. Overhanging Beam.** If the end portion of a beam is extended beyond the support, such beam is known as overhanging beam. Overhanging beam is shown in Fig. 6.3.

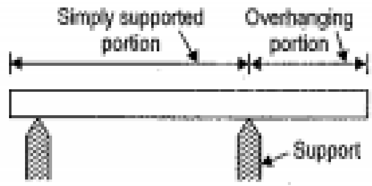



Fig. 6.3

Fig. 6.4

**6.3.4. Fixed Beams.** A beam whose both ends are fixed or built-in walls, is known as fixed beam. Such beam is shown in Fig. 6.4. A fixed beam is also known as a *built-in* or encastred beam.

**6.3.5. Continuous Beam.** A beam which is provided more than two supports as shown in Fig. 6.5, is known as continuous beam.

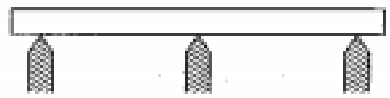


Fig. 6.5

**6.4. TYPES OF LOAD**

A beam is normally horizontal and the loads acting on the beams are generally vertical. The following are the important types of load acting on a beam :

1. Concentrated or point load,
2. Uniformly distributed load, and
3. Uniformly varying load.

**6.4.1. Concentrated or Point Load.** A concentrated load is one which is considered to act at a point, although in practice it must really be distributed over a small area. In Fig. 6.6, $W$ shows the point load.

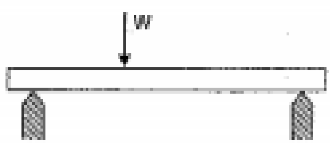



Fig. 6.6

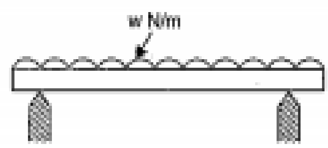



Fig. 6.7

**6.4.2. Uniformly Distributed Load.** A uniformly distributed load is one which is spread over a beam in such a manner that rate of loading $w$ is uniform along the length (*i.e.*, each unit length is loaded to the same rate) as shown in Fig. 6.7. The rate of loading is expressed as $w$ N/m run. Uniformly distributed load is, represented by u.d.l.

For solving the numerical problems, the total uniformly distributed load is converted into a point load, acting at the centre of uniformly distributed load.

**6.4.3. Uniformly Varying Load.** A uniformly varying load is one which is spread over a beam in such a manner that rate of loading varies from point to point along the beam as shown in Fig. 6.8 in which load is zero at one end and increases uniformly to the other end. Such load is known as triangular load.

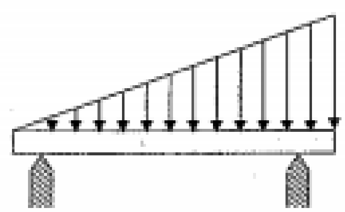


Fig. 6.8

For solving numerical problems the total load is equal to the area of the triangle and this total load is assumed to be acting at the C.G. of the triangle *i.e.*, at a distance of $\frac{2}{3}$rd of total length of beam from left end.

## 6.5. SIGN CONVENTIONS FOR SHEAR FORCE AND BENDING MOMENT

(*i*) *Shear force.* Fig. 6.9 shows a simply supported beam *AB*, carrying a load of 1000 N at its middle point. The reactions at the supports will be equal to 500 N. Hence $R_A = R_B$ = 500 N.

Now imagine the beam to be divided into two portions by the section *X-X*. The resultant of the load and reaction to the left of *X-X* is 500 N vertically upwards. (Note in this case, there is no load to the left of *X-X*). And the resultant of the load and reaction to the right of *X-X* is (1000 ↓ − 500 ↑ = 500 ↓ N) 500 N downwards. The resultant force acting on any one of the parts normal to the axis of the beam is called the *shear force* at the section *X-X*. Here the shear force at the section *X-X* is 500 N.

The shear force at a section will be considered positive when the resultant of the forces to the left to the section is upwards, or to the right of the section is downwards. Similarly the shear force at a section will be considered negative if the resultant of the forces to the left of the section is downwards, or to the right of the section is upwards. Here the resultant force to the left of the section is upwards and hence the shear force will be positive.

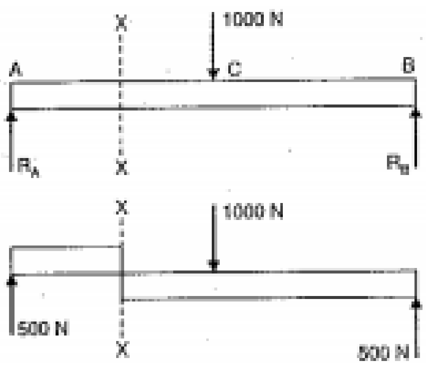



Fig. 6.9

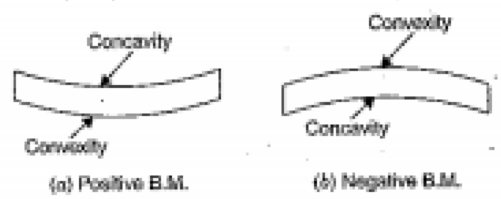



Fig. 6.10

(*ii*) *Bending moment.* The bending moment at a section is considered positive if the bending moment at that section is such that it tends to bend the beam to a curvature having concavity at the top as shown in Fig. 6.10 (*a*). Similarly the bending moment (B.M.) at a section is considered negative if the bending moment at that section is such that it tends to bend the beam to a curvature having convexity at the top as shown in Fig. 6.10 (*b*). The positive B.M. is often called sagging moment and negative B.M. as hogging moment.

Consider the simply supported beam *AB*, carrying a load of 1000 N at its middle point. Reactions $R_A$ and $R_B$ are equal and are having magnitude 500 N as shown in Fig. 6.11. Imagine the beam to be divided into two portions by the section *X-X*. Let the section *X-X* is at a distance of 1 m from A.

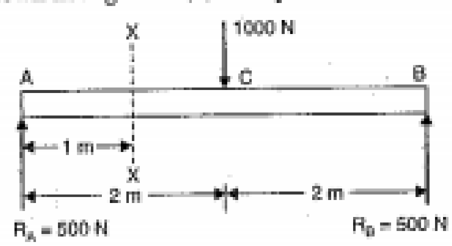



Fig. 6.11

The moments of all the forces (*i.e.*, load and reaction) to the left of $X$-$X$ at the section $X$-$X$ is $R_A \times 1 = 500 \times 1 = 500$ Nm (clockwise). Also the moments of all the forces (*i.e.*, load and reaction) to the right of $X$-$X$ at the section $X$-$X$ is $R_B \times 3$ (anti-clockwise) – $1000 \times 1$ (clockwise) $= 500 \times 3$ Nm – $1000 \times 1$ Nm = 1500 – 1000 = 500 Nm (anti-clockwise).

Hence the tendency of the bending moment at $X$-$X$ is to bend the beam so as to produce concavity at the top as shown in Fig. 6.12.

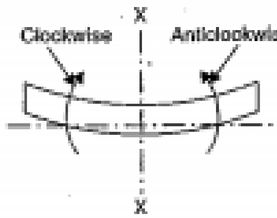



Fig. 6.12

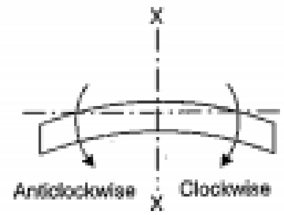



Fig. 6.13

The bending moment at a section is the algebraic sum of the moments of forces and reactions acting on one side of the section. Hence bending moment at the section $X$-$X$ is 500 Nm.

The bending moment will be considered positive when the moment of the forces and reaction on the left portion is clockwise, and on the right portion anti-clockwise. In Fig. 6.12, the bending moment at the section $X$-$X$ is positive.

Similarly the bending moment will be considered negative when the moment of the forces and reactions on the left portion is anti-clockwise, and on the right portion clockwise as shown in Fig. 6.13. In Fig. 6.13, the bending moment at the section $X$-$X$ is negative.

## 6.6. IMPORTANT POINTS FOR DRAWING SHEAR FORCE AND BENDING MOMENT DIAGRAMS

In Art. 6.2, it is mentioned that the shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which show the variation of the bending moment along the length of beam. In these diagrams, the shear force or bending moment are represented by ordinates whereas the length of the beam represents abscissa.

The following are the important points for drawing shear force and bending moment diagrams :

1. Consider the left or the right portion of the section.

2. Add the forces (including reaction) normal to the beam on one of the portion. If right portion of the section is chosen, a force on the right portion acting downwards is positive while a force acting upwards is negative.

If the left portion of the section is chosen, a force on the left portion acting upwards is positive while a force acting downwards is negative.

3. The positive values of shear force and bending moments are plotted above the base line, and negative values below the base line.

4. The shear force diagram will increase or decrease suddenly *i.e.*, by a vertical straight line at a section where there is a vertical point load.

5. The shear force between any two vertical loads will be constant and hence the shear force diagram between two vertical loads will be horizontal.

6. The bending moment at the two supports of a simply supported beam and at the free end of a cantilever will be zero.

## 6.7. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER WITH A POINT LOAD AT THE FREE END

Fig. 6.14 shows a cantilever $AB$ of length $L$ fixed at $A$ and free at $B$ and carrying a point load $W$ at the free end $B$.

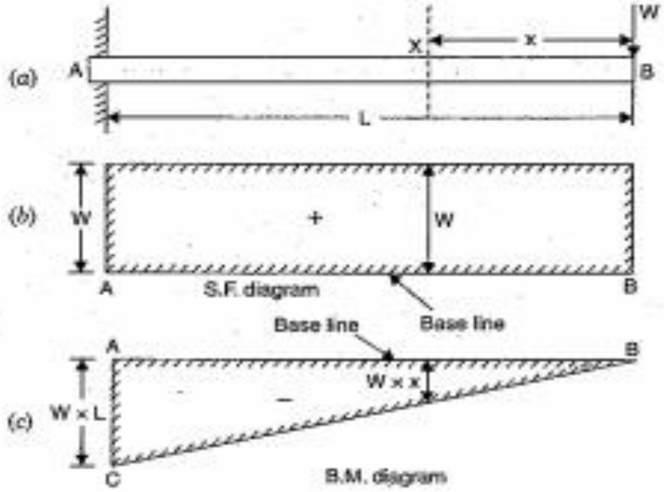



Fig. 6.14

Let $F_x$ = Shear force at $X$, and

$M_x$ = Bending moment at $X$.

Take a section $X$ at a distance $x$ from the free end. Consider the right portion of the section.

The shear force at this section is equal to the resultant force acting on the right portion at the given section. But the resultant force acting on the right portion at the section $X$ is $W$ and acting in the downward direction. But a force on the right portion acting downwards is considered positive. Hence shear force at $X$ is positive.

$$\therefore \quad F_x = +W$$

The shear force will be constant at all sections of the cantilever between $A$ and $B$ as there is no other load between $A$ and $B$. The shear force diagram is shown in Fig. 6.14 ($b$).

*Bending Moment Diagram*

The bending moment at the section $X$ is given by

$$M_x = -W \times x \quad \ldots(i)$$

(Bending moment will be negative as for the right portion of the section, the moment of $W$ at $X$ is clockwise. Also the bending of cantilever will take place in such a manner that convexity will be at the top of the beam).

From equation (*i*), it is clear that B.M. at any section is proportional to the distance of the section from the free end.

At $x = 0$ *i.e.*, at $B$, B.M. = 0

At $x = L$ *i.e.*, at $A$, B.M. = $W \times L$

Hence B.M. follows the straight line law. The B.M. diagram is shown in Fig. 6.14 (*c*). At point $A$, take $AC = W \times L$ in the downward direction. Join point $B$ to $C$.

The shear force and bending moment diagrams for several concentrated loads acting on a cantilever, will be drawn in the similar manner.

**Problem 6.1.** *A cantilever beam of length 2 m carries the point loads as shown in Fig. 6.15. Draw the shear force and B.M. diagrams for the cantilever beam.*

**Sol.** Given :

Refer to Fig. 6.15.

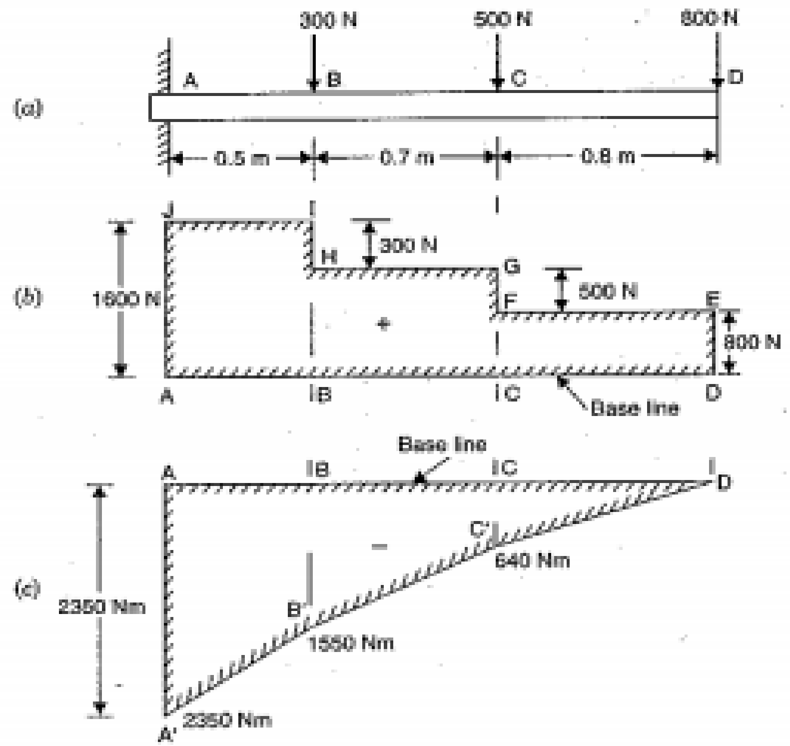



Fig. 6.15

*Shear Force Diagram*

The shear force at $D$ is + 800 N. This shear force remains constant between $D$ and $C$. At $C$, due to point load, the shear force becomes (800 + 500) = 1300 N. Between $C$ and $B$, the shear force remains 1300 N. At $B$ again, the shear force becomes (1300 + 300) = 1600 N. The shear force between $B$ and $A$ remains constant and equal to 1600 N. Hence the shear force at different points will be as given below :

S.F. at $D$, $F_D = +800$ N

S.F. at $C$, $F_C = +800 + 500 = +1300$ N

S.F. at $B$, $F_B = +800 + 500 + 300 = 1600$ N

S.F. at $A$, $F_A = +1600$ N.

The shear force, diagram is shown in Fig. 6.15 ($b$) which is drawn as :

Draw a horizontal line $AD$ as base line. On the base line mark the points $B$ and $C$ below the point loads. Take the ordinate $DE = 800$ N in the upward direction. Draw a line $EF$ parallel to $AD$. The point $F$ is vertically above $C$. Take vertical line $FG = 500$ N. Through $G$, draw a horizontal line $GH$ in which point $H$ is vertically above $B$. Draw vertical line $HI = 300$ N. From $I$, draw a horizontal line $IJ$. The point $J$ is vertically above $A$. This completes the shear force diagram.

*Bending Moment Diagram*

The bending moment at $D$ is zero :

(*i*) The bending moment at any section between $C$ and $D$ at a distance $x$ and $D$ is given by,

$$M_x = -800 \times x \text{ which follows a straight line law.}$$

At $C$, the value of $x = 0.8$ m.

$\therefore$ B.M. at $C$, $M_C = -800 \times 0.8 = -640$ Nm.

(*ii*) The B.M. at any section between $B$ and $C$ at a distance $x$ from $D$ is given by (At $C$, $x = 0.8$ and at $B$, $x = 0.8 + 0.7 = 1.5$ m. Hence here $x$ varies from 0.8 to 1.5).

$$M_x = -800x - 500(x - 0.8) \quad \ldots(i)$$

Bending moment between $B$ and $C$ also varies by a straight line law.

B.M. at $B$ is obtained by substituting $x = 1.5$ m in equation (*i*),

$$\therefore \quad M_B = -800 \times 1.5 - 500(1.5 - 0.8)$$
$$= -1200 - 350 = -1550 \text{ Nm.}$$

(*iii*) The B.M. at any section between $A$ and $B$ at a distance $x$ from $D$ is given by (At $B$, $x = 1.5$ and at $A$, $x = 2.0$ m. Hence here $x$ varies from 1.5 m to 2.0 m)

$$M_x = -800x - 500(x - 0.8) - 300(x - 1.5) \quad \ldots(ii)$$

Bending moment between $A$ and $B$ varies by a straight line law.

B.M. at $A$ is obtained by substituting $x = 2.0$ m in equation (*ii*),

$$\therefore \quad M_A = -800 \times 2 - 500(2 - 0.8) - 300(2 - 1.5)$$
$$= -800 \times 2 - 500 \times 1.2 - 300 \times 0.5$$
$$= -1600 - 600 - 150 = -2350 \text{ Nm.}$$

Hence the bending moments at different points will be as given below :

$$M_D = 0$$
$$M_C = -640 \text{ Nm}$$
$$M_B = -1550 \text{ Nm}$$

and $M_A = -2350$ Nm.

The bending moment diagram is shown in Fig. 6.15 ($c$) which is drawn as.

Draw a horizontal line $AD$ as a base line and mark the points $B$ and $C$ on this line. Take vertical lines $CC' = 640$ Nm, $BB' = 1550$ Nm and $AA' = 2350$ Nm in the downward direction. Join points $D$, $C'$, $B'$ and $A'$ by straight lines. This completes the bending moment diagram.

## 6.8. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

Fig. 6.16 shows a cantilever of length $L$ fixed at $A$ and carrying a uniformly distributed load of $w$ per unit length over the entire length of the cantilever.

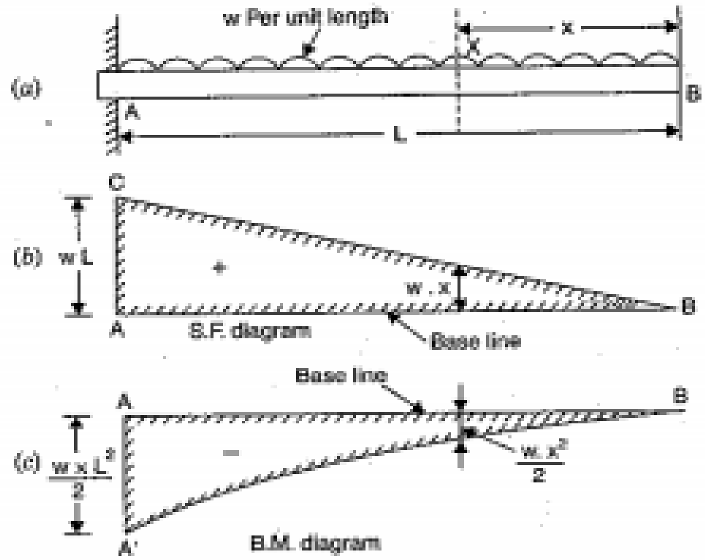



Fig. 6.16

Take a section $X$ at a distance of $x$ from the free end $B$.

Let $F_x$ = Shear force at $X$, and

$M_x$ = Bending moment at $X$.

Here we have considered the right portion of the section. The shear force at the section $X$ will be equal to the resultant force acting on the right portion of the section. But the resultant force on the right portion = $w$ × Length of right portion = $w.x$.

This resultant force is acting downwards. But the resultant force on the right portion acting downwards is considered positive. Hence shear force at $X$ is positive.

$\therefore \quad F_x = +\, w.x$

The above equation shows that the shear force follows a straight line law.

At $B$, $x = 0$ and hence $\quad F_x = 0$

At $A$, $x = L$ and hence $\quad F_x = w.L$

The shear force diagram is shown in Fig. 6.16 (*b*).

*Bending Moment Diagram*

It is mentioned in Art. 6.4.3 that the uniformly distributed load over a section is converted into point load acting at the C.G. of the section.

The bending moment at the section $X$ is given by

$$M_x = -\,(\text{Total load on right portion}) \times \text{Distance of C.G. of right portion from } X$$

$$= -(w\,.\,x)\,.\,\frac{x}{2} = -\,w\,.\,x\,.\,\frac{x}{2} = -\,w\,.\,\frac{x^2}{2} \qquad \ldots(i)$$

(The bending moment will be negative as for the right portion of the section, the moment of the load at $x$ is clockwise. Also the bending of cantilever will take place in such a manner that convexity will be at the top of the cantilever).

From equation ($i$), it is clear that B.M. at any section is proportional to the square of the distance of the section from the free end. This follows a parabolic law.

At $B$, $x = 0$ hence $M_x = 0$

At $A$, $x = L$ hence $M_x = -w \cdot \frac{L^2}{2}$.

The bending moment diagram is shown in Fig. 6.16 ($c$).

**Problem 6.2.** *A cantilever of length 2.0 m carries a uniformly distributed load of 1 kN/m run over a length of 1.5 m from the free end. Draw the shear force and bending moment diagrams for the cantilever.*

**Sol.** Given :

U.D.L., $w = 1$ kN/m run

Refer to Fig. 6.17.

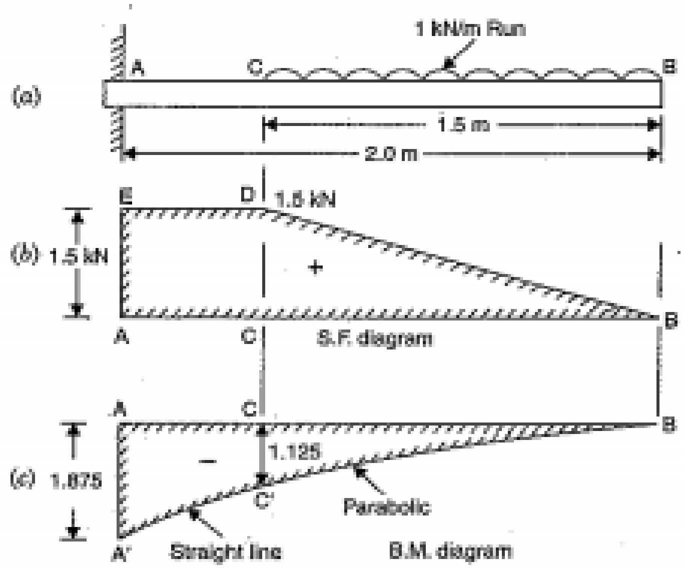



Fig. 6.17

*Shear Force Diagram*

Consider any section between $C$ and $B$ a distance of $x$ from the free end $B$. The shear force at the section is given by

$F_x = w.x$ (+ve sign is due to downward force on right portion of the section)

$= 1.0 \times x$ ($\because$ $w = 1.0$ kN/m run)

At $B$, $x = 0$ hence $F_x = 0$

At $C$, $x = 1.5$ hence $F_x = 1.0 \times 1.5 = 1.5$ kN.

The shear force follows a straight line law between $C$ and $B$. As between $A$ and $C$ there is no load, the shear force will remain constant. Hence shear force between $A$ and $C$ will be represented by a horizontal line.

The shear force diagram is shown in Fig. 6.17 (*b*) in which

$$F_B = 0,\ F_C = 1.5 \text{ kN and } F_A = F_C = 1.5 \text{ kN}.$$

*Bending Moment Diagram*

(*i*) The bending moment at any section between *C* and *B* at a distance *x* from the free end *B* is given by

$$M_x = -(w.x.).\frac{x}{2} = -\left(1.\frac{x^2}{2}\right) = -\frac{x^2}{2} \quad ...(i)$$

(The bending moment will be negative as for the right portion of the section the moment of load at *x* is clockwise).

At *B*, $x = 0$ hence $\quad M_B = -\frac{0^2}{2} = 0$

At *C*, $x = 1.5$ hence $\quad M_C = -\frac{1.5^2}{2} = -1.125$ Nm

From equation (*i*) it is clear that the bending moment varies according to parabolic law between *C* and *B*.

(*ii*) The bending moment at any section between *A* and *C* at a distance *x* from the free end *B* is obtained as : (here *x* varies from 1.5 m to 2.0 m)

Total load due to U.D.L. $= w \times 1.5 = 1.5$ kN.

This load is acting at a distance of $\frac{1.5}{2} = 0.75$ m from the free end *B* or at a distance of $(x - 0.75)$ from any section between *A* and *C*.

∴ Moment of this load at any section between *A* and *C* at a distance *x* from free end

= (Load due to U.D.L.) × $(x - 0.75)$

$$\therefore \quad M_x = -1.5 \times (x - 0.75) \quad ...(ii)$$

(– ve sign is due to clockwise moment for right portion)

From equation (*ii*) it is clear that the bending moment follows straight line law between *A* and *C*.

At *C*, $x = 1.5$ m hence $\quad M_C = -1.5\,(1.5 - 0.75) = -1.125$ Nm

At *A*, $x = 2.0$ m hence $\quad M_A = -1.5\,(2 - 0.75) = -1.875$ Nm.

Now the bending moment diagram is drawn as shown in Fig. 6.17 (*c*). In this diagram line *CC'* = 1.125 Nm and *AA'* = 1.875 Nm. The points *B* and *C'* are on a parabolic curve whereas the points *A'* and *C'* are joined by a straight line.

**Problem 6.3.** *A cantilever of length 2.0 m carries a uniformly distributed load of 2 kN/m length over the whole length and a point load of 3 kN at the free end. Draw the S.F. and B.M. diagrams for the cantilever.*

**Sol.** Given :

Length, $\quad L = 2.0$ m

U.D.L., $\quad w = 2$ kN/m length

Point load at free end $\quad = 3$ kN

Refer to Fig. 6.18.

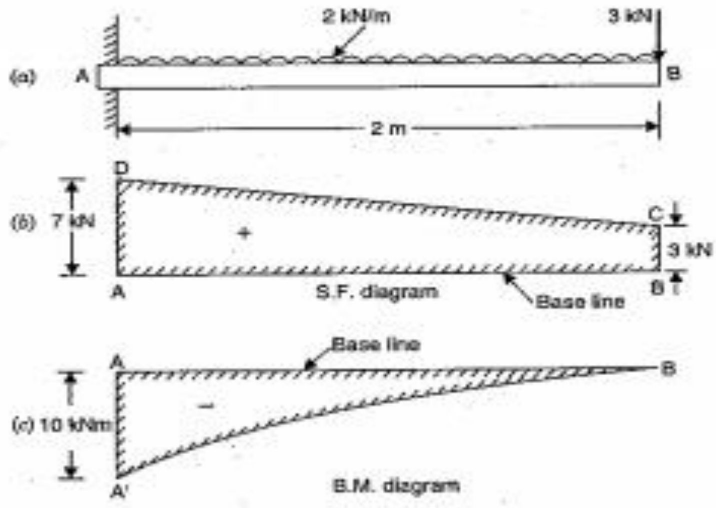



Fig. 6.18

*Shear Force Diagram*

The shear force at $B = 3$ kN

Consider any section at a distance $x$ from the free end $B$. The shear force at the section is given by,

$$F_x = 3.0 + w.x \quad \text{(+ve sign is due to downward force on right portion of the section)}$$

$$= 3.0 + 2 \times x \quad (\because \ w = 2 \text{ kN/m})$$

The above equation shows that shear force follows a straight line law.

At $B$, $x = 0$ hence $F_B = 3.0$ kN

At $A$, $x = 2$ m hence $F_A = 3 + 2 \times 2 = 7$ kN.

The shear force diagram is shown in Fig. 6.18 (*b*) in which $F_B = BC = 3$ kN and $F_A = AD = 7$ kN. The points $C$ and $D$ are joined by a straight line.

*Bending Moment Diagram*

The bending moment at any section at a distance $x$ from the free end $B$ is given by,

$$M_x = -\left(3x + wx \cdot \frac{x}{2}\right)$$

$$= -\left(3x + \frac{2x^2}{2}\right) \quad (\because \ w = 2 \text{ kN/m})$$

$$= -(3x + x^2) \quad \ldots(i)$$

(The bending moment will be negative as for the right portion of the section, the moment of loads at $x$ is clockwise).

The equation (*i*) shows that the B.M. varies according to the parabolic law. From equation (*i*), we have

At $B$, $x = 0$ hence $M_B = -(3 \times 0 + 0^2) = 0$

At $A$, $x = 2$ m hence $M_A = -(3 \times 2 + 2^2) = -10$ kN/m

Now the bending moment diagram is drawn as shown in Fig. 6.18 (*c*). In this diagram, $AA' = 10$ kNm and points $A'$ and $B$ are joined by a parabolic curve.

**Problem 6.8.** *A simply supported beam of length 6 m, carries point load of 3 kN and 6 kN at distances of 2 m and 4 m from the left end. Draw the shear force and bending moment diagrams for the beam.*

**Sol.** First calculate the reactions $R_A$ and $R_B$.

Taking moments of the force about $A$, we get

$$R_B \times 6 = 3 \times 2 + 6 \times 4 = 30$$

$$\therefore \quad R_B = \frac{30}{6} = 5 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load on beam} - R_B = (3 + 6) - 5 = 4 \text{ kN}$$

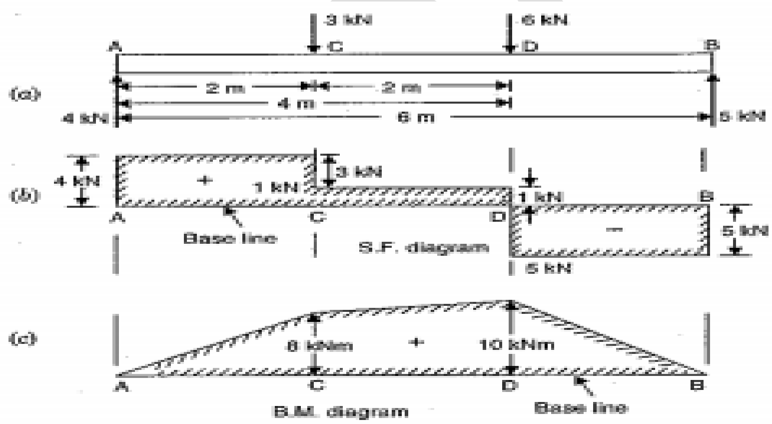



Fig. 6.26

*Shear Force Diagram*

Shear force at $A$, $\quad F_A = + R_A = + 4$ kN

Shear force between $A$ and $C$ is constant and equal to + 4 kN

Shear force at $C$, $\quad F_C = + 4 - 3.0 = + 1$ kN

Shear force between $C$ and $D$ is constant and equal to + 1 kN.

Shear force at $D$, $\quad F_D = + 1 - 6 = - 5$ kN

The shear force between $D$ and $B$ is constant and equal to − 5 kN.

Shear force at $B$, $\quad F_B = - 5$ kN

The shear force diagram is drawn as shown in Fig. 6.26 (*b*).

*Bending Moment Diagram*

B.M. at $A$, $\quad M_A = 0$

B.M. at $C$, $\quad M_C = R_A \times 2 = 4 \times 2 = + 8$ kNm

B.M. at $D$, $\quad M_D = R_A \times 4 - 3 \times 2 = 4 \times 4 - 3 \times 2 = + 10$ kNm

B.M. at $B$, $\quad M_B = 0$

The bending moment diagram is drawn as shown in Fig. 6.26 (*c*).

### 6.12. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A SIMPLY SUPPORTED BEAM CARRYING A UNIFORMLY DISTRIBUTED LOAD

Fig. 6.27 shows a beam $AB$ of length $L$ simply supported at the ends $A$ and $B$ and carrying a uniformly distributed load of $w$ per unit length over the entire length. The reactions at the supports will be equal and their magnitude will be half the total load on the entire length.

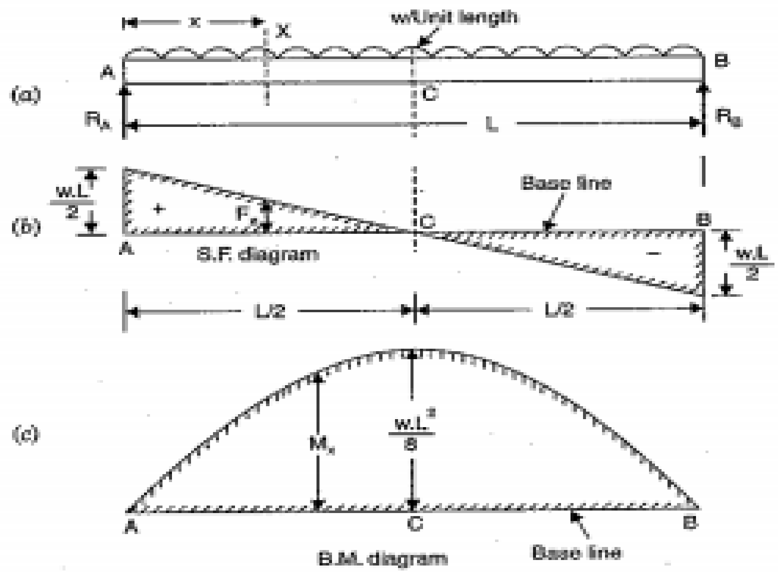



Fig. 6.27

Let $R_A$ = Reaction at $A$, and

$R_B$ = Reaction at $B$

$\therefore \quad R_A = R_B = \frac{w.L}{2}$

Consider any section $X$ at a distance $x$ from the *left end A*. The shear force at the section (*i.e.*, $F_x$) is given by,

$$F_x = +R_A - w\,.\,x = +\frac{w.L}{2} - w\,.\,x \quad ...(i)$$

From equation (*i*), it is clear that the shear force varies according to straight line law. The values of shear force at different points are :

At $A$, $x = 0$ hence $\quad F_A = +\frac{w.L}{2} - \frac{w.0}{2} = +\frac{w.L}{2}$

At $B$, $x = L$ hence $\quad F_B = +\frac{w.L}{2} - w.L = -\frac{w.L}{2}$

At $C$, $x = \frac{L}{2}$ hence $\quad F_C = +\frac{w.L}{2} - w.\frac{L}{2} = 0$

The shear force diagram is drawn as shown in Fig. 6.27 (*b*).

The bending moment at the section $X$ at a distance $x$ from left end $A$ is given by,

$$M_x = +R_A\,.\,x - w\,.\,x\,.\,\frac{x}{2}$$

$$= \frac{w.L}{2}\,.\,x - \frac{w.x^2}{2} \qquad \left(\because\ R_A = \frac{w.L}{2}\right) \quad ...(ii)$$

From equation (*ii*), it is clear that B.M. varies according to parabolic law.

The values of B.M. at different points are :

At $A$, $x = 0$ hence $M_A = \frac{w.L}{2}.0 - \frac{w.0}{2} = 0$

At $B$, $x = L$ hence $M_B = \frac{w.L}{2}.L - \frac{w}{2}.L^2 = 0$

At $C$, $x = \frac{L}{2}$ hence $M_C = \frac{w.L}{2}.\frac{L}{2} - \frac{w}{2}.\left(\frac{L}{2}\right)^2 = \frac{w.L^2}{4} - \frac{w.L^2}{8} = +\frac{w.L^2}{8}$.

Thus the B.M. increases according to parabolic law from zero at $A$ to $+\frac{w.L^2}{8}$ at the middle point of the beam and from this value the B.M. decreases to zero at $B$ according to the parabolic law.

Now the B.M. diagram is drawn as shown in Fig. 6.27 (*c*).

**Problem 6.9.** *Draw the shear force and bending moment diagram for a simply supported beam of length 9 m and carrying a uniformly distributed load of 10 kN/m for a distance of 6 m from the left end. Also calculate the maximum B.M. on the section.*

**Sol.** First calculate reactions $R_A$ and $R_B$.

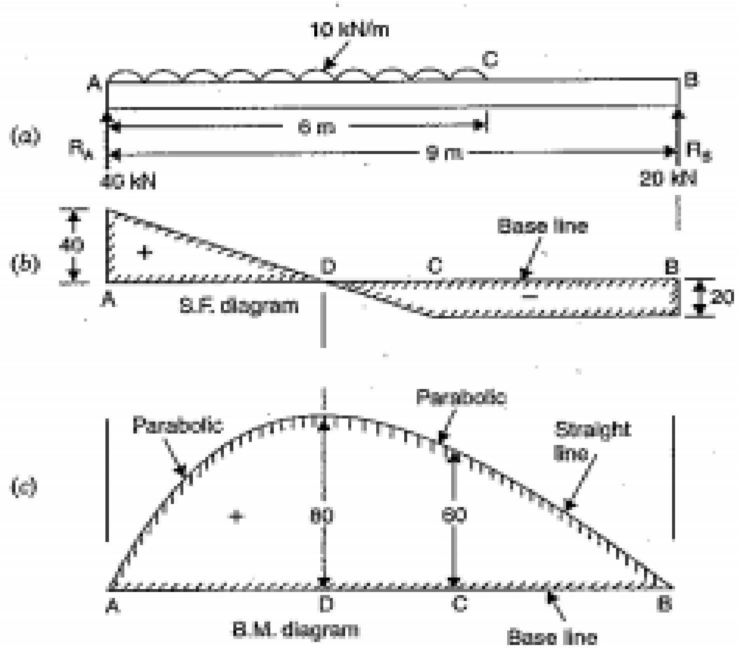



Fig. 6.28

Taking moments of the forces about $A$, we get

$$R_B \times 9 = 10 \times 6 \times \frac{6}{2} = 180$$

$\therefore$ $$R_B = \frac{180}{9} = 20 \text{ kN}$$

$\therefore$ $$R_A = \text{Total load on beam} - R_B = 10 \times 6 - 20 = 40 \text{ kN}.$$

*Shear Force Diagram*

Consider any section at a distance $x$ from $A$ between $A$ and $C$. The shear force at the section is given by,

$$F_x = +R_A - 10\,x = +40 - 10\,x \qquad ...(i)$$

Equation (*i*) shows that shear force varies by a straight line law between $A$ and $C$.

At $A$, $x = 0$ hence $\qquad F_A = +40 - 0 = 40$ kN

At $C$, $x = 6$ m hence $\qquad F_C = +40 - 10 \times 6 = -20$ kN

The shear force at $A$ is + 40 kN and at $C$ is – 20 kN. Also shear force between $A$ and $C$ varies by a straight line. This means that somewhere between $A$ and $C$, the shear force is zero. Let the S.F. is zero at $x$ metre from $A$. Then substituting the value of S.F. (*i.e.*, $F_x$) equal to zero in equation (*i*), we get

$$0 = 40 - 10x$$

$$\therefore \qquad x = \frac{40}{10} = 4 \text{ m}$$

Hence shear force is zero at a distance 4 m from $A$.

The shear force is constant between $C$ and $B$. This equal to – 20 kN.

Now the shear force diagram is drawn as shown in Fig. 6.28 (*b*). In the shear force diagram, distance $AD = 4$ m. The point $D$ is at a distance 4 m from $A$.

*B.M. Diagram*

The B.M. at any section between $A$ and $C$ at a distance $x$ from $A$ is given by,

$$M_x = R_A \times x - 10\,.\,x\,.\,\frac{x}{2} = 40x - 5x^2 \qquad ...(ii)$$

Equation (*ii*) shows that B.M. varies according to parabolic law between $A$ and $C$.

At $A$, $x = 0$ hence $\qquad M_A = 40 \times 0 - 5 \times 0 = 0$

At $C$, $x = 6$ m hence $\qquad M_C = 40 \times 6 - 5 \times 6^2 = 240 - 180 = +60$ kNm

At $D$, $x = 4$ m hence $\qquad M_D = 40 \times 4 - 5 \times 4^2 = 160 - 80 = +80$ kNm

The bending moment between $C$ and $B$ varies according to linear law.

B.M. at $B$ is zero whereas at $C$ is 60 kNm.

The bending moment diagram is drawn as shown in Fig. 6.28 (*c*).

*Maximum Bending Moment*

The B.M. is maximum at a point where shear force changes sign. This means that the point where shear force becomes zero from positive value to the negative or *vice-versa*, the B.M. at that point will be maximum. From the shear force diagram, we know that at point $D$, the shear force is zero after changing its sign. Hence B.M. is maximum at point $D$. But the B.M. at $D$ is + 80 kNm.

$\therefore$ Max. B.M. = **+ 80 kN. Ans.**

**Problem 6.10.** *Draw the shear force and B.M. diagrams for a simply supported beam of length 8 m and carrying a uniformly distributed load of 10 kN/m for a distance of 4 m as shown in Fig. 6.29.*

**Sol.** First calculate the reactions $R_A$ and $R_B$.

Taking moments of the forces about $A$, we get

$$R_B \times 8 = 10 \times 4 \times \left(1 + \frac{4}{2}\right) = 120$$

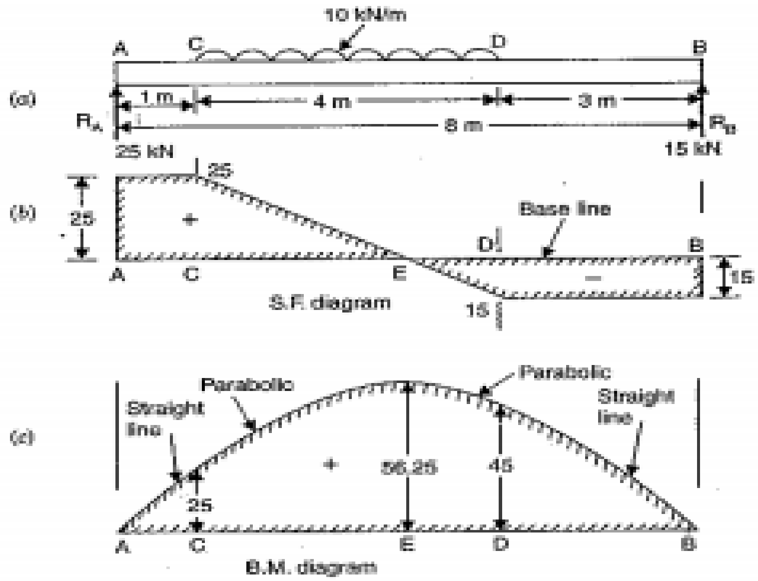



Fig. 6.29

$$\therefore \quad R_B = \frac{120}{8} = 15 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load on beam} - R_B$$
$$= 10 \times 4 - 15 = 25 \text{ kN}$$

*Shear Force Diagram*

The shear force at $A$ is + 25 kN. The shear force remains constant between $A$ and $C$ and equal to + 25 kN. The shear force at $B$ is − 15 kN. The shear force remains constant between $B$ and $D$ and equal to − 15 kN. The shear force at any section between $C$ and $D$ at a distance $x$ from $A$ is given by,

$$F_x = +25 - 10(x - 1) \quad \text{...(i)}$$

At $C$, $x = 1$ hence $\quad F_C = +25 - 10(1 - 1) = +25$ kN

At $D$, $x = 5$ hence $\quad F_D = +25 - 10(5 - 1) = -15$ kN

The shear force at $C$ is + 25 kN and at $D$ is − 15 kN. Also shear force between $C$ and $D$ varies by a straight line law. This means that somewhere between $C$ and $D$, the shear force is zero. Let the S.F. be zero at $x$ metre from $A$. Then substituting the value of S.F. (*i.e.*, $F_x$) equal to zero in equation (*i*), we get

$$0 = 25 - 10(x - 1)$$

or $\quad 0 = 25 - 10x + 10 \quad$ or $\quad 10x = 35$

$$\therefore \quad x = \frac{35}{10} = 3.5 \text{ m}$$

Hence the shear force is zero at a distance 3.5 m from $A$.

Hence the distance $AE$ = 3.5 m in the shear force diagram shown in Fig. 6.29 (*b*).

*B.M. Diagram*

B.M. at $A$ is zero

B.M. at $B$ is also zero

B.M. at $C = R_A \times 1 = 25 \times 1 = 25$ kNm

The B.M. at any section between $C$ and $D$ at a distance $x$ from $A$ is given by,

$$M_x = R_A \cdot x - 10(x-1) \cdot \frac{(x-1)}{2} = 25 \times x - 5(x-1)^2 \qquad \text{...(ii)}$$

At $C$, $x = 1$ hence $M_C = 25 \times 1 - 5(1-1)^2 = 25$ kNm

At $D$, $x = 5$ hence $M_D = 25 \times 5 - 5(5-1)^2 = 125 - 80 = 45$ kNm

At $E$, $x = 3.5$ hence $M_E = 25 \times 3.5 - 5(3.5-1)^2 = 87.5 - 31.25 = 56.25$ kNm

B.M. will increase from 0 at $A$ to 25 kNm at $C$ by a straight line law. Between $C$ and $D$ the B.M. varies according to parabolic law as is clear from equation (*ii*). Between $C$ and $D$, the B.M. will be maximum at $E$. From $D$ to $B$ the B.M. will decrease from 45 kNm at $D$ to zero at $B$ according to straight line law.

**Problem 6.11.** *Draw the S.F. and B.M. diagrams of a simply supported beam of length 7 m carrying uniformly distributed loads as shown in Fig. 6.30.*

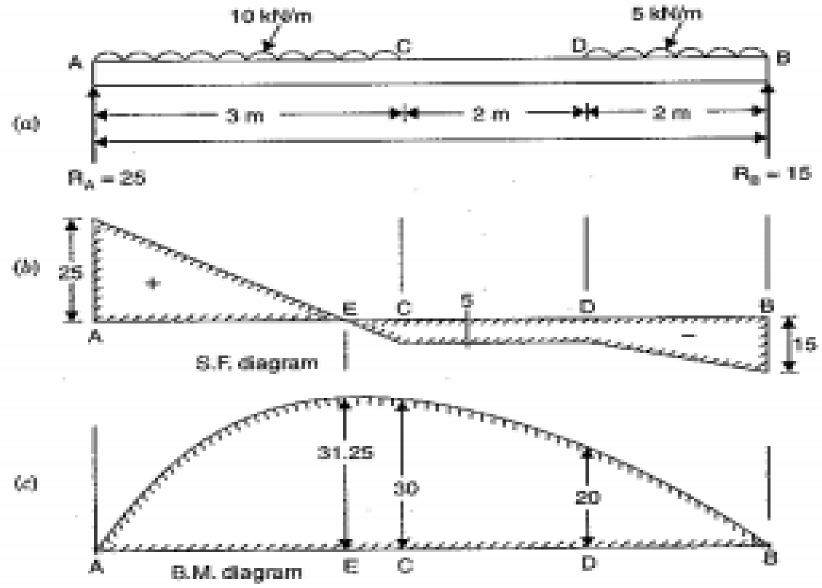



Fig. 6.30

**Sol.** First calculate the reactions $R_A$ and $R_B$.

Taking moments of all forces about $A$, we get

$$R_B \times 7 = 10 \times 3 \times \frac{3}{2} + 5 \times 2 \times \left(3 + 2 + \frac{2}{2}\right) = 45 + 60 = 105$$

$$\therefore \qquad R_B = \frac{105}{7} = 15 \text{ kN}$$

and $R_A$ = Total load on beam − $R_B$

$= (10 \times 3 + 5 \times 2) - 15 = 40 - 15 = 25$ kN

*S.F. Diagram*

The shear force at $A$ is + 25 kN

The shear force at $C = R_A - 3 \times 10 = +25 - 30 = -5$ kN

The shear force varies between $A$ and $C$ by a straight line law.

The shear force between $C$ and $D$ is constant and equal to − 5 kN

The shear force at $B$ is − 15 kN

The shear force between $D$ and $B$ varies by a straight line law.

The shear force diagram is drawn as shown in Fig. 6.30 (*b*).

The shear force is zero at point $E$ between $A$ and $C$. Let us find the location of $E$ from $A$. Let the point $E$ be at a distance $x$ from $A$.

The shear force at $E = R_A - 10 \times x = 25 - 10x$

But shear force at $E = 0$

$\therefore \quad 25 - 10x = 0 \quad$ or $\quad 10x = 25$

or
$$x = \frac{25}{10} = 2.5 \text{ m}$$

*B.M. Diagram*

B.M. at $A$ is zero

B.M. at $B$ is zero

B.M. at $C$, $\quad M_C = R_A \times 3 - 10 \times 3 \times \frac{3}{2} = 25 \times 3 - 45 = 75 - 45 = 30$ kNm

At $E$, $x = 2.5$ and hence

B.M. at $E$, $\quad M_E = R_A \times 2.5 - 10 \times 2.5 \times \frac{2.5}{2} = 25 \times 2.5 - 5 \times 6.25$

$= 62.5 - 31.25 = 31.25$ kNm

B.M. at $D$, $\quad M_D = 25(3 + 2) - 10 \times 3 \times \left(\frac{3}{2} + 2\right) = 125 - 105 = 20$ kNm

The B.M. between $AC$ and between $BD$ varies according to parabolic law. But B.M. between $C$ and $D$ varies according to straight line law. Now the bending moment diagram is drawn as shown in Fig. 6.30 (*c*).

**Problem 6.12.** *A simply supported beam of length 10 m, carries the uniformly distributed load and two point loads as shown in Fig. 6.31. Draw the S.F. and B.M. diagram for the beam. Also calculate the maximum bending moment.*

**Sol.** First calculate the reactions $R_A$ and $R_B$.

Taking moments of all forces about $A$, we get

$$R_B \times 10 = 50 \times 2 + 10 \times 4 \times \left(2 + \frac{4}{2}\right) + 40(2 + 4)$$

$$= 100 + 160 + 240 = 500$$

$$\therefore \quad R_B = \frac{500}{10} = 50 \text{ kN}$$

and $R_A$ = Total load on beam − $R_B$

$= (50 + 10 \times 4 + 40) - 50 = 130 - 50 = 80$ kN

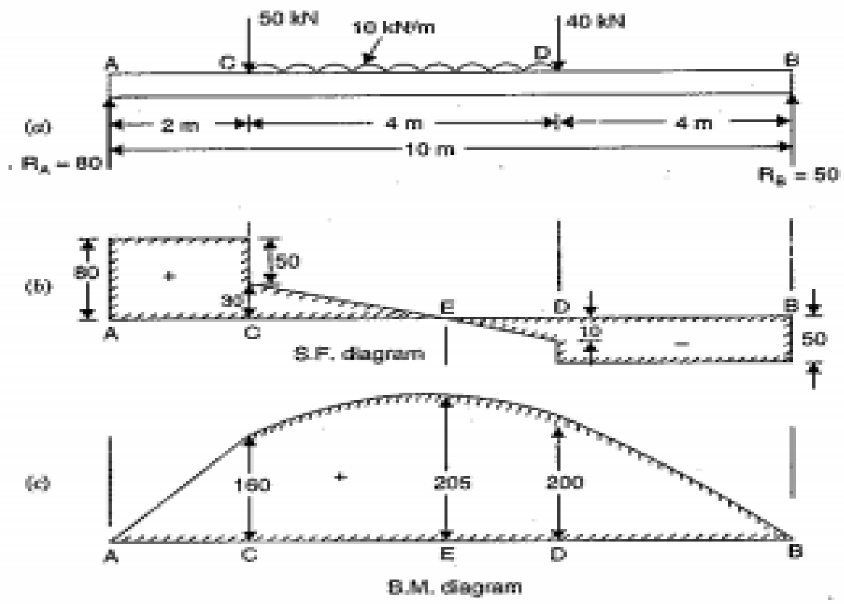



Fig. 6.31

*S.F. Diagram*

The S.F. at $A$, $F_A = R_A = +80$ kN

The S.F. will remain constant between $A$ and $C$ and equal to + 80 kN

The S.F. just on R.H.S. of $C = R_A - 50 = 80 - 50 = 30$ kN

The S.F. just on L.H.S. of $D = R_A - 50 - 10 \times 4 = 80 - 50 - 40 = -10$ kN

The S.F. between $C$ and $D$ varies according to straight line law.

The S.F. just on R.H.S. of $D = R_A - 50 - 10 \times 4 - 40 = 80 - 50 - 40 - 40 = -50$ kN

The S.F. at $B = -50$ kN

The S.F. remains constant between $D$ and $B$ and equal to $-50$ kN

The shear force diagram is drawn as shown in Fig. 6.31 (*b*).

The shear force is zero at point $E$ between $C$ and $D$.

Let the distance of $E$ from point $A$ is $x$.

Now shear force at $E = R_A - 50 - 10 \times (x - 2)$

$= 80 - 50 - 10x + 20 = 50 - 10x$

But shear force at $E = 0$

$$\therefore \quad 50 - 10x = 0 \quad \text{or} \quad x = \frac{50}{10} = 5 \text{ m}$$

*B.M. Diagram*

B.M. at $A$ is zero

B.M. at $B$ is zero

B.M. at $C$, $M_C = R_A \times 2 = 80 \times 2 = 160$ kNm

B.M. at $D$, $M_D = R_A \times 6 - 50 \times 4 - 10 \times 4 \times \frac{4}{2}$

$= 80 \times 6 - 200 - 80 = 480 - 200 - 80 = 200$ kNm

At $E$, $x = 5$ m and hence B.M. at $E$,

$$M_E = F_A \times 5 - 50(5-2) - 10 \times (5-2) \times \left(\frac{5-2}{2}\right)$$

$$= 80 \times 5 - 50 \times 3 - 10 \times 3 \times \frac{3}{2} = 400 - 150 - 45 = 205 \text{ kNm}$$

The B.M. between $C$ and $D$ varies according to parabolic law reaching a maximum value at $E$. The B.M. between $A$ and $C$ and also between $B$ and $D$ varies according to linear law. The B.M. diagram is shown in Fig. 6.31 (c).

*Maximum B.M.*

The maximum B.M. is at $E$, where S.F. becomes zero after changing its sign.

$\therefore$ Max. B.M. = $M_E$ = **205 kNm. Ans.**

**Problem 6.15.** *Draw the S.F. and B.M. diagrams for the overhanging beam carrying uniformly distributed load of 2 kN/m over the entire length and a point load of 2 kN as shown in Fig. 6.36. Locate the point of contraflexure.*

**Sol.** First calculate the reactions $R_A$ and $R_B$.

Taking moments of all forces about $A$, we get

$$R_B \times 4 = 2 \times 6 \times 3 + 2 \times 6 = 36 + 12 = 48$$

$$\therefore \quad R_B = \frac{48}{4} = 12 \text{ kN}$$

and $R_A$ = Total load $- R_B = (2 \times 6 + 2) - 12 = 2$ kN

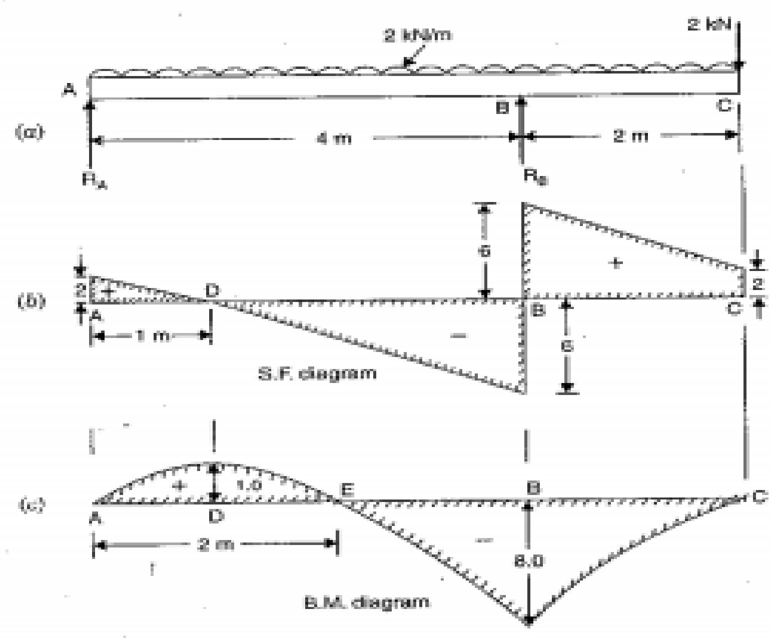



Fig. 6.36

*S.F. Diagram*

S.F. at $A = +R_A = +2$ kN

(*i*) The S.F. at any section between *A* and *B* at a distance *x* from *A* is given by,

$$F_x = +R_A - 2 \times x$$
$$= 2 - 2x \qquad ...(i)$$

At *A*, $x = 0$ hence $\quad F_A = 2 - 2 \times 0 = 2$ kN

At *B*, $x = 4$ hence $\quad F_A = 2 - 2 \times 4 = -6$ kN

The S.F. between *A* and *B* varies according to straight line law. At *A*, S.F. is positive and at *B*, S.F. is negative. Hence between *A* and *B*, S.F. is zero. The point of zero S.F. is obtained by subetituting $F_x = 0$ in equation (*i*).

$$\therefore \quad 0 = 2 - 2x \quad \text{or} \quad x = \frac{2}{2} = 1 \text{ m}$$

The S.F. is zero at point *D*. Hence distance of *D* from *A* is 1 m.

(*ii*) The S.F. at any section between *B* and *C* at a distance *x* from *A* is given by,

$$F_x = +R_A - 2 \times 4 + R_B - 2(x - 4)$$
$$= 2 - 8 + 12 - 2(x - 4) = 6 - 2(x - 4) \qquad ...(ii)$$

At *B*, $x = 4$ hence $\quad F_B = 6 - 2(4 - 4) = +6$ kN

At *C*, $x = 6$ hence $\quad F_C = 6 - 2(6 - 4) = 6 - 4 = 2$ kN

The S.F. diagram is drawn as shown in Fig. 6.36 (*b*).

*B.M. Diagram*

B.M. at *A* is zero

(*i*) B.M. at any section between *A* and *B* at a distance *x* from *A* is given by,

$$M_x = R_A \times x - 2 \times x \times \frac{x}{2} = 2x - x^2 \qquad ...(iii)$$

The above equation shows that the B.M. between *A* and *B* varies according to parabolic law.

At *A*, $x = 0$ hence $\quad M_A = 0$

At *B*, $x = 4$ hence $\quad M_B = 2 \times 4 - 4^2 = -8$ kNm

Max. B.M. is at *D* where S.F. is zero after changing sign

At *D*, $x = 1$ hence $\quad M_D = 2 \times 1 - 1^2 = 1$ kNm

The B.M. at *C* is zero. The B.M. also varies between *B* and *C* according to parabolic law. Now the B.M. diagram is drawn as shown in Fig. 6.36 (*c*).

*Point of Contraflexure*

This point is at *E* between *A* and *B*, where B.M. is zero after changing its sign. The distance of *E* from *A* is obtained by putting $M_x = 0$ in equation (*iii*).

$$\therefore \quad 0 = 2x - x^2 = x(2 - x)$$
$$2 - x = 0$$

and $\quad x = 2$ m. **Ans.**

# MODULE 6

## SPRINGS & COLUMNS

### 16.14. SPRINGS

Springs are the elastic bodies which absorb energy due to resilience. The absorbed energy may be released as and when required. A spring which is capable of absorbing the greatest amount of energy for the given stress, without getting permanently distorted, is known as the best spring. The two important types of springs are :

1. Laminated or leaf springs and
2. Helical springs.

**16.14.1. Laminated or leaf spring.** The laminated springs are used to absorb shocks in railway wagons, coaches and road vehicles (such as cars, lorries etc.).

Fig. 16.11 shows a laminated spring which consists of a number of parallel strips of a metal having different lengths and same width, placed one over the other. Initially all the plates are bent to the same radius and are free to slide one over the other. Fig. 16.11 shows the initial position of the spring, which is having some central deflection δ. The spring rests on the axis of the vehicle and its top plate is pinned at the ends to the chassis of the vehicle.

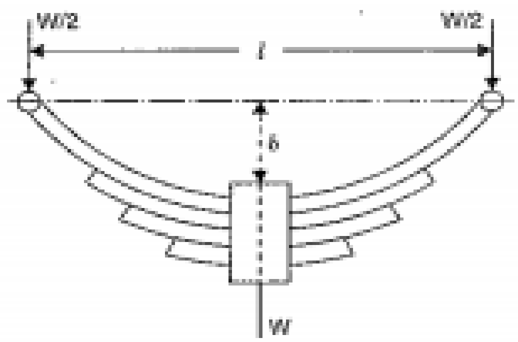


Fig. 16.11

When the spring is loaded to the designed load $W$, all the plates becomes flat and the central deflection (δ) disappears.

Let $b$ = Width of each plate

$n$ = Number of plates

$l$ = Span of spring

$\sigma$ = Maximum bending stress developed in the plates

$t$ = Thickness of each plates

$W$ = Point load acting at the centre of the spring and

$\delta$ = Original deflection of the top spring.

**Expression for maximum bending stress developed in the plate.** The load $W$ acting at the centre of the lowermost plate, will be shared equally on the two ends of the top plate as shown in Fig. 16.11.

$\therefore$ B.M. at the centre = Load at one end × $\frac{l}{2}$

or
$$M = \frac{W}{2} \times \frac{l}{2} = \frac{W \cdot l}{4} \quad \ldots(i)$$

The moment of inertia of each plate,

$$I = \frac{bt^3}{12}$$

But the relation among bending stress ($\sigma$), bending moment ($M$) and moment of inertia ($I$) is given by

$$\frac{M}{I} = \frac{\sigma}{y} \qquad \left(\text{Here } y = \frac{t}{2}\right)$$

or

$$M = \frac{\sigma}{y} \times I = \frac{\sigma \times \frac{bt^3}{12}}{\frac{t}{2}} = \frac{\sigma \, . \, bt^2}{6}$$

$\therefore$ Total resisting moment by $n$ plates

$$= n \times M = \frac{n \times \sigma \, . \, bt^2}{6} \qquad \text{...(ii)}$$

As the maximum B.M. due to load is equal to the total resisting moment, therefore equating (*i*) and (*ii*),

$$\frac{W \, . \, l}{4} = \frac{n\sigma \, . \, bt^2}{6}$$

$$\therefore \quad \sigma = \frac{6W \, . \, l}{4 \, . \, n \, . \, b \, . \, t^2} = \frac{3Wl}{2nbt^2} \qquad \text{...(16.22)}$$

Equation (16.22) gives the maximum stress developed in the plate of the spring.

**Expression for central deflection of the leaf spring**

Now $R$ = Radius of the plate to which they are bent.

From triangle $ACO$ of Fig. 16.12, we have

$$AO^2 = AC^2 + CO^2$$

or

$$R^2 = \left(\frac{l}{2}\right)^2 + (R - \delta)^2$$

$$= \frac{l^2}{4} + R^2 + \delta^2 - 2R\delta$$

$$= \frac{l^2}{4} + R^2 - 2R\delta \quad \text{(Neglecting } \delta^2 \text{ which is a small quantity)}$$

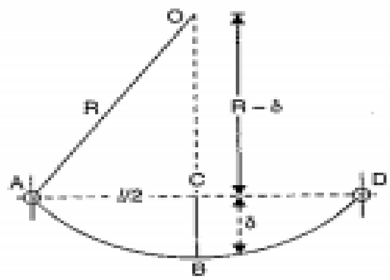



Fig. 16.12

$$\therefore \quad 2R\delta = \frac{l^2}{4}$$

$$\therefore \quad \delta = \frac{l^2}{4 \times 2R} = \frac{l^2}{8R} \qquad \text{...(iii)}$$

But the relation between bending stress, modulus of elasticity and radius of curvature ($R$) is given by

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$R = \frac{E \times y}{\sigma} = \frac{E \times t}{2\sigma} \qquad \left(\text{Here } y = \frac{t}{2}\right)$$

Substituting this value of $R$ in equation (*iii*), we get

$$\delta = \frac{l^2 \times 2\sigma}{8 \times E \times t} = \frac{\sigma . l^2}{4Et} \quad ...(16.23)$$

Equation (16.23) gives the central deflection of the spring.

**Problem 16.33.** *A leaf spring carries a central load of 3000 N. The leaf spring is to be made of 10 steel plates 5 cm wide and 6 mm thick. If the bending stress is limited to 150 N/mm² determine :*

*(i) Length of the spring and*

*(ii) Deflection at the centre of the spring.*

*Take $E = 2 \times 10^5$ N/mm².*

**Sol.** Given :

Central load, $W = 3000$ N

No. of plates, $n = 10$

Width of each plates, $b = 5$ cm $= 50$ mm

Thickness, $t = 6$ mm

Bending stress, $\sigma = 150$ N/mm²

Modulus of elasticity, $E = 2 \times 10^5$ N/mm².

Let $l$ = Length of spring

$\delta$ = Deflection at the centre of spring.

Using equation (16.22),

$$\sigma = \frac{3Wl}{2nbt^2}$$

or

$$150 = \frac{3 \times 3000 \times l}{2 \times 10 \times 50 \times 6^2}$$

$$\therefore \quad l = \frac{150 \times 2 \times 10 \times 50 \times 6^2}{3 \times 3000} = \textbf{600 mm.} \quad \textbf{Ans.}$$

Using equation (16.23) for deflection,

$$\delta = \frac{\sigma . l^2}{4Et} = \frac{150 \times 600^2}{4 \times 2 \times 10^5 \times 6} = \textbf{11.25 mm.} \quad \textbf{Ans.}$$

**Problem 16.34.** *A laminated spring 1 m long is made up of plates each 5 cm wide and 1 cm thick. If the bending stress in the plate is limited to 100 N/mm², how many plates would be required to enable the spring to carry a central point load of 2 kN ? If $E = 2.1 \times 10^5$ N/mm², what is the deflection under the load ?* (AMIE, Summer 1982)

**Sol.** Given :

Length of spring, $l = 1$ m $= 1000$ mm

Width of each plate, $b = 5$ cm $= 50$ mm

Thickness of each plate, $t = 1$ cm $= 10$ mm

Bending stress, $\sigma = 100$ N/mm²

Central load on spring, $W = 2$ kN $= 2000$ N

Young's modulus, $E = 2.1 \times 10^5$ N/mm²

Let $n$ = Number of plates and

$\delta$ = Deflection under the load.

Using the equation (16.22),

$$\sigma = \frac{3Wl}{2nbt^2} \quad \text{or} \quad 100 = \frac{3 \times 2000 \times 1000}{2 \times n \times 50 \times 10^2}$$

or

$$n = \frac{3 \times 2000 \times 1000}{100 \times 2 \times 50 \times 100} = 6. \quad \textbf{Ans.}$$

*Deflection under load*

Using equation (16.23),

$$\delta = \frac{\sigma \times l^2}{4E \times t} = \frac{100 \times 1000^2}{4 \times 2.1 \times 10^5 \times 10} = \mathbf{11.9\ mm.} \quad \textbf{Ans.}$$

**16.14.2. Helical Springs.** Helical springs are the thick spring wires coiled into a helix. They are of two types :

1. Close-coiled helical springs and
2. Open coiled helical springs.

**Close-coiled helical springs.** Close-coiled helical springs are the springs in which helix angle is very small or in other words the pitch between two adjacent turns is small. A close-coiled helical spring carrying an axial load is shown in Fig. 16.13. As the helix angle in case of close-coiled helical springs are small, hence the bending effect on the spring is ignored and we assume that the coils of a close-coiled helical springs are to stand purely torsional stresses.

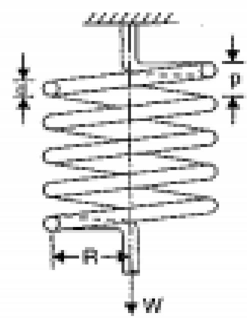


Fig. 16.13

**Expression for max. shear stress induced in wire.** Fig. 16.13 shows a close-coiled helical spring subjected to an axial load.

Let $d$ = Diameter of spring wire
$p$ = Pitch of the helical spring
$n$ = Number of coils
$R$ = Mean radius of spring coil
$W$ = Axial load on spring
$C$ = Modulus of rigidity
$\tau$ = Max. shear stress induced in the wire
$\theta$ = Angle of twist in spring wire, and
$\delta$ = Deflection of spring due to axial load
$l$ = Length of wire.

Now twisting moment on the wire,

$$T = W \times R \qquad ...(i)$$

But twisting moment is also given by

$$T = \frac{\pi}{16} \tau d^3 \qquad ...(ii)$$

Equating equations (*i*) and (*ii*), we get

$$W \times R = \frac{\pi}{16} \tau d^3 \quad \text{or} \quad \tau = \frac{16W \times R}{\pi d^3} \qquad ...(16.24)$$

Equation (16.24) gives the max. shear stress induced in the wire.

**Expression for deflection of spring**

Now length of one coil = $\pi D$ or $2\pi R$

$\therefore$ Total length of the wire = Length of one coil × No. of coils or $l = 2\pi R \times n$.

As the every section of the wire is subjected to torsion, hence the strain energy stored by the spring due to torsion is given by equation (16.20).

$\therefore$ Strain energy stored by the spring,

$$U = \frac{\tau^2}{4C} \cdot \text{Volume} = \frac{\tau^2}{4C} \cdot \text{Volume}$$

$$= \left(\frac{16W.R}{\pi d^3}\right)^2 \times \frac{1}{4C} \times \left(\frac{\pi}{4} d^2 \times 2\pi R.n\right)$$

$$\left(\because \tau = \frac{16WR}{\pi d^3} \text{ and Volume} = \frac{\pi}{4} d^2 \times \text{Total length of wire}\right)$$

$$= \frac{32W^2R^2}{Cd^4} .R.n = \frac{32W^2R^3 . n}{Cd^4} \quad \text{...(16.25)}$$

Work done on the spring = Average load × Deflection

$$= \tfrac{1}{2} W \times \delta \quad (\because \text{ Deflection} = \delta)$$

Equating the work done on spring to the energy stored, we get

$$\tfrac{1}{2} W.\delta = \frac{32W^2R^3 . n}{Cd^4}$$

$$\therefore \quad \delta = \frac{64WR^3n}{Cd^4} \quad \text{...(16.26)}$$

**Expression for stiffness of spring**

The stiffness of spring,

$$s = \text{Load per unit deflection}$$

$$= \frac{W}{\delta} = \frac{W}{\dfrac{64 . WR^3 . n}{Cd^4}} = \frac{Cd^4}{64 . R^3 . n} \quad \text{...(16.27)}$$

**Note.** The solid length of the spring means the distance between the coils when the coils are touching each other. There is no gap between the coils. The solid length is given by

$$\text{Solid length} = \text{Number of coils} \times \text{Dia. of wire} = n \times d \quad \text{...(16.28)}$$

**Problem 16.35.** *A closely coiled helical spring is to carry a load of 500 N. Its mean coil diameter is to be 10 times that of the wire diameter. Calculate these diameters if the maximum shear stress in the material of the spring is to be 80 N/mm².*

(AMIE, Summer 1985)

**Sol.** Given :

Load on spring, $W = 500$ N

Max. shear stress, $\tau = 80 \text{ N/mm}^2$

Let $d$ = Diameter of wire

$D$ = Mean diameter of coil

$\therefore$ $D = 10\,d$.

Using equation (16.24), $\tau = \dfrac{16WR}{\pi d^3}$

or $$80 = \frac{16 \times 500 \times \left(\frac{D}{2}\right)}{\pi d^3} \qquad \left(\because \; R = \frac{D}{2}\right)$$

$$= \frac{8000 \times \left(\frac{10d}{2}\right)}{\pi d^3} \qquad (\because \; D = 10d)$$

or $$80 \times \pi d^3 = 8000 \times 5d$$

or $$d^2 = \frac{8000 \times 5}{80 \times \pi} = 159.25$$

$\therefore$ $$d = \sqrt{159.15} = 12.6 \text{ mm} = \mathbf{1.26 \text{ cm.}} \quad \textbf{Ans.}$$

$\therefore$ $$D = 10 \times d = 10 \times 1.26 = \mathbf{12.6 \text{ cm.}} \quad \textbf{Ans.}$$

**Problem 16.36.** *In problem 16.35, if the stiffness of the spring is 20 N per mm deflection and modulus of rigidity = 8.6 × 10⁴ N/mm², find the number of coils in the closely coiled helical spring.*

**Sol.** Given :

Stiffness, $s = 20$ N/mm

Modulus of rigidity, $C = 8.4 \times 10^4$ N/mm$^2$

From problem 16.35,

$W = 500$ N, $\tau = 80$ N/mm$^2$

$d = 12.6$ mm and D = 126 mm

$\therefore$ $R = D/2 = 126/2 = 63$ mm

Let $n$ = Number of coils in the spring

We know, stiffness $= \dfrac{\text{Load}}{\delta}$

or $$20 = \frac{500}{\delta}$$

$\therefore$ $$\delta = \frac{500}{20} = 25 \text{ mm}$$

Using equation (16.26),

$$\delta = \frac{65WR^3 \cdot n}{C.d^4}$$

or $$25 = \frac{64 \times 500 \times (63)^3 \times n}{8.4 \times 10^4 \times 12.6^4} \qquad (\because \; R = 63 \text{ mm})$$

$\therefore$ $$n = \frac{25 \times 8.4 \times 10^4 \times 12.6^4}{64 \times 500 \times (63)^3} = 6.6 \text{ say } 7.0$$

or $$n = \mathbf{7.} \quad \textbf{Ans.}$$

## 19.1. INTRODUCTION

Column or strut is defined as a member of a structure, which is subjected to axial compressive load. If the member of the structure is vertical and both of its ends are fixed rigidly while subjected to axial compressive load, the member is known as *column*, for example a vertical pillar between the roof and floor. If the member of the structure is not vertical and one or both of its ends are hinged or pin joined, the bar is known as *strut*. Examples of struts are : connecting rods, piston rods etc.

## 19.2. FAILURE OF A COLUMN

The failure of a column takes place due to the anyone of the following stresses set up in the columns :

(*i*) Direct compressive stresses,

(*ii*) Buckling stresses, and

(*iii*) Combined of direct compressive and buckling stresses.

**19.2.1. Failure of a Short Column.** A short column of uniform cross-sectional area $A$, subjected to an axial compressive load $P$, is shown in Fig. 19.1. The compressive stress induced is given by

$$p = \frac{P}{A}$$

If the compressive load on the short column is gradually increased, a stage will reach when the column will be on the point of failure by crushing. The stress induced in the column corresponding to this load is known as crushing stress and the load is called crushing load.

Let $P_c$ = Crushing load,

$\sigma_c$ = Crushing stress, and

$A$ = Area of cross-section.

Then $$\sigma_c = \frac{P_c}{A}.$$

All short columns fail due to crushing.

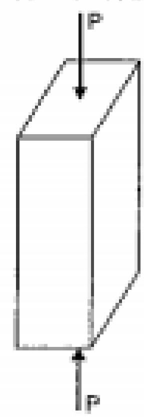



Fig. 19.1

**19.2.2. Failure of a Long Column.** A long column of uniform cross-sectional area $A$ and of length $l$, subjected to an axial compressive load $P$, is shown in Fig. 19.2. A column is known as long column if the length of the column in comparison to its lateral dimensions, is very large. Such columns do not fail by crushing alone, but also by bending (also known buckling) as shown in Fig. 19.2. The load at which the column just buckles, is known as *buckling load or critical just or crippling load*. The buckling load is less than the crushing load for a long column. Actually the value of buckling load for long columns is low whereas for short columns the value of buckling load is relatively high.

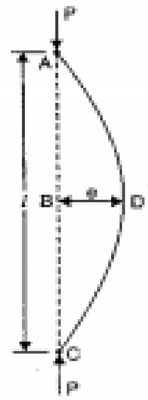


Fig. 19.2

Refer to Fig. 19.2.

Let $l$ = Length of a long column

$P$ = Load (compressive) at which the column has just buckled

$A$ = Cross-sectional area of the column

$e$ = Maximum bending of the column at the centre

$\sigma_0$ = Stress due to direct load $= \frac{P}{A}$

$\sigma_b$ = Stress due to bending at the centre of the column $= \frac{P \times e}{Z}$

where $Z$ = Section modulus about the axis of bending.

The extreme stresses on the mid-section are given by

Maximum stress $= \sigma_0 + \sigma_b$

and Minimum stress $= \sigma_0 - \sigma_b$.

The column will fail when maximum stress (*i.e.*, $\sigma_0 + \sigma_b$) is more than the crushing stress $\sigma_c$. But in case of long columns, the direct compressive stresses are negligible as compared to buckling stresses. Hence very long columns are subjected to buckling stresses only.

## 19.3. ASSUMPTIONS MADE IN THE EULER'S COLUMN THEORY

The following assumptions are made in the Euler's column theory :

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

## 19.4. END CONDITIONS FOR LONG COLUMNS

In case of long columns, the stress due to direct load is very small in comparison with the stress due to buckling. Hence the failure of long columns take place entirely due to buckling (or bending). The following four types of end conditions of the columns are important :

1. Both the ends of the column are hinged (or pinned).
2. One end is fixed and the other end is free.
3. Both the ends of the column are fixed.
4. One end is fixed and the other is pinned.

For a hinged end, the deflection is zero. For a fixed end the deflection and slope are zero. For a free end the deflection is not zero.

**19.4.1. Sign Conventions.** The following sign conventions for the bending of the columns will be used :

1. A moment which will bend the column with its *convexity* towards its initial central line as shown in Fig. 19.3 (*a*) is taken as positive. In Fig. 19.3 (*a*), *AB* represents the initial centre line of a column. Whether the column bends taking the shape *AB'* or *AB"*, the moment producing this type of curvature is positive.

2. A moment which will tend to bend the column with its *concavity* towards its initial centre line as shown in Fig. 19.3 (*b*) is taken as negative.

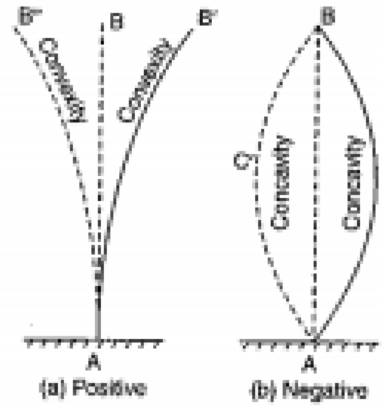



Fig. 19.3

## 19.5. EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE HINGED

The load at which the column just buckles (or bends) is called crippling load. Consider a column *AB* of length *l* and uniform cross-sectional area, hinged at both of its ends *A* and *B*. Let *P* be the crippling load at which the column has just buckled. Due to the crippling load, the column will deflect into a curved form *ACB* as shown in Fig. 19.4.

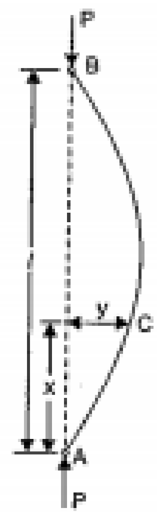



Fig. 19.4

Consider any section at a distance $x$ from the end $A$.

Let $y$ = Deflection (lateral displacement) at the section.

The moment due to the crippling load at the section $= -P \cdot y$

(– ve sign is taken due to sign convention given in Art. 19.4.1)

But moment $= EI \dfrac{d^2y}{dx^2}$ .

Equating the two moments, we have

$$EI \frac{d^2y}{dx^2} = -P \cdot y \quad \text{or} \quad EI \frac{d^2y}{dx^2} + P \cdot y = 0$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right) \quad \text{...}(i)$$

where $C_1$ and $C_2$ are the constants of integration. The values of $C_1$ and $C_2$ are obtained as given below :

(*i*) At *A*, $x = 0$ and $y = 0$ (See Fig. 19.4)

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos 0^\circ + C_2 \sin 0$$

$$= C_1 \times 1 + C_2 \times 0 \qquad (\because \cos 0 = 1 \text{ and } \sin 0 = 0)$$

$$= C_1$$

$$\therefore \quad C_1 = 0. \qquad ...(ii)$$

(*ii*) At *B*, $x = l$ and $y = 0$ (See Fig. 19.4).

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos\left(l \times \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right)$$

$$= 0 + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right) \qquad [\because C_1 = 0 \text{ from equation } (ii)]$$

$$= C_2 \sin\left(l \sqrt{\frac{P}{EI}}\right) \qquad ...(iii)$$

From equation (*iii*), it is clear that either $C_2 = 0$

or $$\sin\left(l \sqrt{\frac{P}{EI}}\right) = 0.$$

As $C_1 = 0$, then if $C_2$ is also equal to zero, then from equation (*i*) we will get $y = 0$. This means that the bending of the column will be zero or the column will not bend at all. Which is not true.

$$\therefore \quad \sin\left(l \sqrt{\frac{P}{EI}}\right) = 0$$

$$= \sin 0 \text{ or } \sin \pi \text{ or } \sin 2\pi \text{ or } \sin 3\pi \text{ or } ...$$

or $$l \sqrt{\frac{P}{EI}} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } ...$$

Taking the least practical value,

$$l \sqrt{\frac{P}{EI}} = \pi$$

or $$P = \frac{\pi^2 EI}{l^2}. \qquad ...(19.1)$$

## 19.9. EFFECTIVE LENGTH (OR EQUIVALENT LENGTH) OF A COLUMN

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

Let $L_e$ = Effective length of a column,

$l$ = Actual length of the column, and

$P$ = Crippling load for the column.

Then the crippling load for any type of end condition is given by

$$P = \frac{\pi^2 EI}{L_e^2}. \qquad ...(19.5)$$

The crippling load (*P*) in terms of actual length and effective length and also the relation between effective length and actual length are given in Table 19.1.

| S.No. | End conditions of column | Crippling load in terms of | | Relation between effective length and actual length |
|---|---|---|---|---|
| | | Actual length | Effective length | |
| 1. | Both ends hinged | $\frac{\pi^2 EI}{l^2}$ | $\frac{\pi^2 EI}{L_e^2}$ | $L_e = l$ |
| 2. | One end is fixed and other is free | $\frac{\pi^2 EI}{4l^2}$ | $\frac{\pi^2 EI}{L_e^2}$ | $L_e = 2l$ |
| 3. | Both ends fixed | $\frac{4\pi^2 EI}{l^2}$ | $\frac{\pi^2 EI}{L_e^2}$ | $L_e = \frac{l}{2}$ |
| 4. | One end fixed and other is hinged | $\frac{4\pi^2 EI}{l^2}$ | $\frac{\pi^2 EI}{L_e^2}$ | $L_e = \frac{l}{\sqrt{2}}$ |

There are two values of moment of inertia *i.e.*, $I_{xx}$ and $I_{yy}$.

The value of $I$ (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

**19.9.2. Slenderness Ratio.** The ratio of the actual length of a column to the least radius of gyration of the column, is known as slenderness ratio.

Mathematically, slenderness ratio is given by

$$\text{Slenderness ratio} = \frac{\text{Actual length}}{\text{Least radius of gyration}} = \frac{l}{k} \qquad \ldots(19.8)$$

## 19.10. LIMITATION OF EULER'S FORMULA

From equation (19.6), we have

$$\text{Crippling stress} = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}$$

For a column with both ends hinged, $L_e = l$. Hence Crippling stress becomes as $= \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$,

where $\frac{l}{k}$ is slenderness ratio.

If the slenderness ratio $\left(i.e., \frac{l}{k}\right)$ is small, the crippling stress (or the stress at failure) will be high. But for the column material, the crippling stress cannot be greater than the crushing stress. Hence when the slenderness ratio is less than a certain limit, Euler's formula gives a value of crippling stress greater than the crushing stress. In the limiting case, we can find the value of $l/k$ for which crippling stress is equal to crushing stress.

For example, for a mid steel column with both ends hinged,

Crushing stress = 330 N/mm²

Young's modulus, $E = 2.1 \times 10^5$ N/mm².

Equating the crippling stress to the crushing stress corresponding to the minimum value of slenderness ratio, we get

Crippling stress = Crushing stress

or
$$\frac{\pi^2 E}{\left(\frac{l}{k}\right)^2} = 330 \quad \text{or} \quad \frac{\pi^2 \times 2.1 \times 10^5}{\left(\frac{l}{k}\right)^2} = 330$$

$$\therefore \quad \left(\frac{l}{k}\right)^2 = \frac{\pi^2 \times 2.1 \times 10^5}{330} = 6282$$

$$\therefore \quad \frac{l}{k} = \sqrt{6282} = 79.27, \text{ say } 80.$$

Hence, if the slenderness ratio is less than 80 for mild steel column with both ends hinged, then Euler's formula will not be valid.

**Problem 19.1.** *A solid round bar 3 m long and 5 cm in diameter is used as a strut with both ends hinged. Determine the crippling (or collapsing) load. Take $E = 2.0 \times 10^5$ N/mm².*

**Sol.** Given :

Length of bar, $l = 3 \text{ m} = 3000 \text{ mm}$

Diameter of bar, $d = 5 \text{ cm} = 50 \text{ mm}$

Young's modulus, $E = 2.0 \times 10^5 \text{ N/mm}^2$

Moment of inertia, $I = \frac{\pi}{64} \times 5^4 = 30.68 \text{ cm}^4 = 30.68 \times 10^4 \text{ mm}^4$

Let $P$ = Crippling load.

As both the ends of the bar are hinged, hence the crippling load is given by equation (19.1).

$$\therefore \quad P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$$

$$= 67288 \text{ N} = \mathbf{67.288 \text{ kN}.} \quad \textbf{Ans.}$$

**Problem 19.2.** *For the problem 19.1 determine the crippling load, when the given strut is used with the following conditions :*

*(i) One end of the strut is fixed and the other end is free*

*(ii) Both the ends of strut are fixed*

*(iii) One end is fixed and other is hinged.*

**Sol.** Given :

The data from Problem 19.1, is $l = 3000$ mm, diameter = 50 mm, $E = 2.0 \times 10^5 \text{ N/mm}^2$ and $I = 30.68 \times 10^4 \text{ mm}^4$.

Let $P$ = Crippling load.

*(i) Crippling load when one end is fixed and other is free*

Using equation (19.2), $P = \frac{\pi^2 EI}{4l^2} = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{4 \times 3000^2} = \mathbf{16822 \text{ N}.}$ **Ans.**

**Alternate Method**

The crippling load for any type of end condition is given by equation (19.5) as,

$$P = \frac{\pi^2 EI}{L_e^2} \quad \ldots(i)$$

where $L_e$ = Effective length.

The effective length ($L_e$) when one end is fixed and other end is free from Table 19.1 on page 819 is given as

$$L_e = 2l = 2 \times 3000 = 6000 \text{ mm}$$

Substituting the value of $L$ in equation (*i*), we get

$$P = \frac{\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{6000^2} = \mathbf{16822 \text{ N}.} \quad \textbf{Ans.}$$

*(ii) Crippling load when both the ends are fixed*

Using equation (19.3), $P = \frac{4\pi^2 EI}{l^2} = \frac{4\pi^2 \times 2 \times 10^5 \times 30.68 \times 10^4}{3000^2}$

$$= 269152 \text{ N} = \mathbf{269.152 \text{ kN}.} \quad \textbf{Ans.}$$

**Alternate Method**

Using equation (19.5), $P = \frac{\pi^2 EI}{L_e^2}$

where $L_e$ = Effective length

$$= \frac{l}{2} \quad \text{(when both the ends are fixed)}$$

$$= \frac{3000}{2} \quad (\because \; l = 3000)$$

$$= 1500 \text{ mm}$$

$$\therefore \quad P = \frac{\pi^2 \times 2.0 \times 10^5 \times 30.68 \times 10^4}{1500^2} = \textbf{269152 N. Ans.}$$

*(iii) Crippling load when one end is fixed and the other is hinged*

Using equation (19.4), $P = \frac{2\pi^2 EI}{l^2} = \frac{2 \times \pi^2 \times 2.0 \times 10^5 \times 30.68 \times 10^4}{3000^2} = \textbf{134576 N. Ans.}$

**Alternate Method**

Using equation (19.5), $P = \frac{\pi^2 EI}{L_e^2}$

where $L_e$ = Effective length

$$= \frac{l}{\sqrt{2}} \quad \text{(when one end is fixed and the other is hinged)}$$

$$= \frac{3000}{\sqrt{2}}$$

$$\therefore \quad P = \frac{\pi^2 \times 2.0 \times 10^5 \times 30.68 \times 10^4}{\left(\frac{3000}{\sqrt{2}}\right)^2} = \textbf{134576 N. Ans.}$$

**Problem 19.3.** *A column of timber section 15 cm × 20 cm is 6 metre long both ends being fixed. If the Young's modulus for timber = 17.5 kN/mm², determine :*

*(i) Crippling load and*

*(ii) Safe load for the column if factor of safety = 3.*

**Sol.** Given :

Dimension of section = 15 cm × 20 cm

Actual length, $l$ = 6 m = 6000 mm

Young's modulus, $E$ = 17.5 kN/mm²

*(i) Let P = Crippling load*

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2} \quad \ldots(i)$$

where $L_e$ = Effective length

$$= \frac{l}{2} \quad \text{(when both the ends are fixed)}$$

$$= \frac{6000}{2} = 3000 \text{ mm} \qquad (\because \; l = 6000 \text{ mm})$$

$I$ = Least value of moment of inertia

Moment of inertia of the section about $X$-$X$ axis,

$$I_{XX} = \frac{15 \times 20^3}{12} = 10000 \text{ cm}^4$$
$$= 10000 \times 10^4 \text{ mm}^4$$

And moment of inertia of the section about $Y$-$Y$ axis,

$$I_{YY} = \frac{20 \times 15^3}{12} = 5625 \text{ cm}^4$$
$$= 5625 \times 10^4 \text{ mm}^4.$$

Since $I_{YY}$ is less than $I_{XX}$, therefore the column will tend to buckle in $Y$-$Y$ direction.

And the value of $I$ will be the least value of the two moment of inertia.

$$\therefore \quad I = 5625 \text{ cm}^4 = 5625 \times 10^4 \text{ mm}^4$$

Substituting the values of $I = 5625 \times 10^4$ mm$^4$ and $L = 3000$ mm in equation (*i*), we get

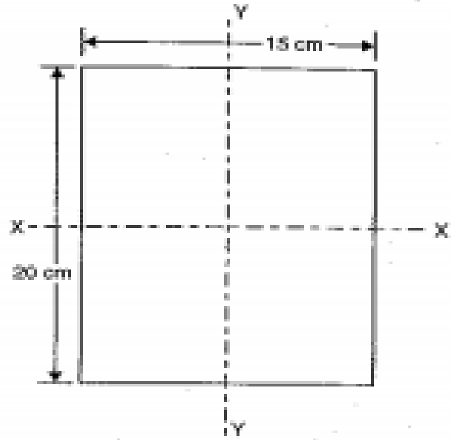



Fig. 19.8

$$P = \frac{\pi^2 \times 17.5 \times 5625 \times 10^4}{3000} = \textbf{1079.48 kN.} \quad \textbf{Ans.}$$

(*ii*) *Safe load for the column*

Factor of safety = 3.0 (given)

$$\therefore \text{ Safe load} = \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{1079.48}{3} = \textbf{359.8 say 360 kN.} \quad \textbf{Ans.}$$

**Problem 19.4.** *A hollow mild steel tube 6 m long 4 cm internal diameter and 6 mm thick is used as a strut with both ends hinged. Find the crippling load and safe load taking factor of safety as 3. Take $E = 2 \times 10^5$ N/mm$^2$.*

**Sol.** Given :

Length of tube, $l = 6 \text{ m} = 600 \text{ cm}$

Internal dia., $d = 4 \text{ cm}$

Thickness, $t = 5 \text{ mm} = 0.5 \text{ cm}$

$\therefore$ External dia., $D = d + 2t = 4 + 2 \times 0.5 = 4 + 1 = 5 \text{ cm}$

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$

Factor of safety = 3.0

$$\text{Moment of inertia of section, } I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}[5^4 - 4^4] \text{ cm}^4$$
$$= \frac{\pi}{64}(625 - 256) = 18.11 \text{ cm}^4 = 18.11 \times 10^4 \text{ mm}^4$$

Since both ends of the strut are hinged,

$\therefore$ Effective length, $L_e = l = 600 \text{ cm} = 6000 \text{ mm}$

Let $P$ = Crippling load

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2}$$
$$= \frac{\pi^2 \times 2.0 \times 10^5 \times 18.11 \times 10^4}{6000^2} = \textbf{9929.9 say 9930 N.} \quad \textbf{Ans.}$$

$$\text{And safe load} = \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{9930}{3.0} = \textbf{3310 N.} \quad \textbf{Ans.}$$

**Problem 19.5.** *A solid round bar 4 m long and 5 cm in diameter was found to extend 4.6 mm under a tensile load of 50 kN. This bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.0.*

**Sol.** Given :

Actual length of bar, $L = 4$ m $= 4000$ mm

Dia. of bar, $d = 5$ cm

$\therefore$ Area of bar, $A = \frac{\pi}{4} \times 5^2 = 6.25\pi \text{ cm}^2 = 6.25\pi \times 10^2 \text{ mm}^2 = 625\pi \text{ mm}^2$

Extension of bar, $\delta L = 4.6$ mm

Tensile load, $W = 50$ kN $= 50000$ N.

In this problem, the values of Young's modulus ($E$) is not given. But it can be calculated from the given data.

$$\therefore \text{ Young's modulus, } E = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{\left(\frac{\text{Tensile load}}{\text{Area}}\right)}{\left(\frac{\text{Extension of bar}}{\text{Length of bar}}\right)}$$

$$\left(\because \text{ Stress} = \frac{\text{Load}}{\text{Area}} \text{ and strain} = \frac{\delta L}{L}\right)$$

$$= \frac{\left(\frac{W}{A}\right)}{\frac{\delta L}{L}} = \frac{W}{A} \times \frac{L}{\delta L} = \frac{50000}{625\,\pi} \times \frac{4000}{4.6} = 2.214 \times 10^4 \text{ N/mm}^2.$$

Since the strut is hinged at its both ends,

$\therefore$ Effective length, $L_e$ = Actual length = 4000 mm

Let $P$ = Crippling or buckling load.

Using equation (19.5), we get

$$P = \frac{\pi^2 EI}{L_e^2}$$

$$= \frac{\pi^2 \times 2.214 \times 10^4 \times \frac{\pi}{64} \times 5^4 \times 10^4}{4000 \times 4000} \qquad \left(\because I = \frac{\pi}{64} \times 5^4 \times 10^4 \text{ mm}^4\right)$$

$= \mathbf{4189.99 \text{ say } 4190 \text{ N.}}$ **Ans.**

And safe load $= \dfrac{\text{Crippling load}}{\text{Factor of safety}} = \dfrac{4190}{4} = \mathbf{1047.5 \text{ N.}}$ **Ans.**

## 19.11. RANKINE'S FORMULA

In Art. 19.10, we have learnt that Euler's formula gives correct results only for very long columns. But what happens when the column is a short or the column is not a very long. On the basis of results of experiments performed by Rankine, he established an empirical formula which is applicable to all columns whether they are short or long. The empirical formula given by Rankine is known as Rankine's formula, which is given as

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} \qquad \ldots(i)$$

where $P$ = Crippling load by Rankine's formula

$P_C$ = Crushing load = $\sigma_c \times A$

$\sigma_c$ = Ultimate crushing stress

$A$ = Area of cross-section

$P_E$ = Crippling load by Euler's formula

$= \frac{\pi^2 EI}{L_e^2}$, in which $L_e$ = Effective length

For a given column material the crushing stress $\sigma_c$ is a constant. Hence the crushing load $P_C$ (which is equal to $\sigma_c \times A$) will also be constant for a given cross-sectional area of the column. In equation (*i*), $P_C$ is constant and hence value of $P$ depends upon the value of $P_E$. But for a given column material and given cross-sectional area, the value of $P_E$ depends upon the effective length of the column.

(*i*) *If the column is a short*, which means the value of $L_e$ is small, then the value of $P_E$ will be large. Hence the value of $\frac{1}{P_E}$ will be small enough and is negligible as compared to the value of $\frac{1}{P_C}$. Neglecting the value of $\frac{1}{P_E}$ in equation (*i*), we get

$$\frac{1}{P} \rightarrow \frac{1}{P_C} \quad \text{or} \quad P \rightarrow P_C.$$

Hence the crippling load by Rankine's formula for a short column is approximately equal to crushing load. In Art. 19.2.1 also we have seen that short columns fail due to crushing.

(*ii*) *If the column is long*, which means the value of $L_e$ is large. Then the value of $P_E$ will be small and the value of $\frac{1}{P_E}$ will be large enough compared with $\frac{1}{P_C}$. Hence the value of $\frac{1}{P_C}$ may be neglected in equation (*i*).

$$\therefore \quad \frac{1}{P} = \frac{1}{P_E} \quad \text{or} \quad P \rightarrow P_E.$$

Hence the crippling load by Rankine's formula for long columns is approximately equal to crippling load given by Euler's formula.

Hence the Rankine's formula $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E}$ gives satisfactory results for all lengths of columns, ranging from short to long columns.

Now the Rankine's formula is $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} = \frac{P_E + P_C}{P_C \cdot P_E}$.

Taking reciprocal to both sides, we have

$$P = \frac{P_C \cdot P_E}{P_E + P_C} = \frac{P_C}{1 + \frac{P_C}{P_E}}$$

$$= \frac{\sigma_c \times A}{1 + \frac{\sigma_c \cdot A}{\left(\frac{\pi^2 EI}{L_e^2}\right)}} \qquad \left(\because \; P_c = \sigma_c \cdot A \text{ and } P_E = \frac{\pi^2 EI}{L_e^2}\right)$$

But $I = Ak^2$, where $k$ = least radius of gyration

$\therefore$ The above equation becomes as

$$P = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c \cdot A \cdot L_e^2}{\pi^2 E \cdot Ak^2}} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c}{\pi^2 E} \cdot \left(\frac{L_e}{k}\right)^2}$$

$$= \frac{\sigma_c \cdot A}{1 + a \cdot \left(\frac{L_e}{k}\right)^2} \qquad \text{...(19.9)}$$

where $a = \frac{\sigma_c}{\pi^2 E}$ and is known as Rankine's constant.

The equation (19.9) gives crippling load by Rankine's formula. As the Rankine formula is empirical formula, the value of '$a$' is taken from the results of the experiments and is not calculated from the values of $\sigma_c$ and $E$.

The values of $\sigma_c$ and $a$ for different columns material are given below in Table 19.2.

**TABLE 19.2**

| *S. No.* | *Material* | $\sigma_c$ *in N/mm²* | $a$ |
|---|---|---|---|
| 1. | Wrought Iron | 250 | $\frac{1}{9000}$ |
| 2. | Cast Iron | 550 | $\frac{1}{1600}$ |
| 3. | Mild Steel | 320 | $\frac{1}{7500}$ |
| 4. | Timber | 50 | $\frac{1}{750}$ |

**Problem 19.13.** *The external and internal diameter of a hollow cast iron column are 5 cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the values of* $\sigma_c = 550$ *N/mm²* *and* $a = \frac{1}{1600}$ *in Rankine's formula.*

**Sol.** Given :

External dia., $D = 5$ cm

Internal dia., $d = 4$ cm

$\therefore$ Area, $A = \frac{\pi}{4}(5^2 - 4^2) = 2.25\pi \text{ cm}^2 = 2.25\pi \times 10^2 \text{ mm}^2 = 225\pi \text{ mm}^2$

Moment of Inertia, $I = \frac{\pi}{64}[5^4 - 4^4] = 5.7656\,\pi \text{ cm}^4$

$= 5.7656\pi \times 10^4 \text{ mm}^4 = 57656\pi \text{ mm}^4$

∴ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\pi}{225\pi}} = 25.625 \text{ mm}$$

Length of column, $l$ = 3 m = 3000 mm

As both the ends are fixed,

∴ Effective length, $L_e = \frac{l}{2} = \frac{3000}{2} = 1500$ mm

Crushing stress, $\sigma_c = 550$ N/mm²

Rankine's constant, $a = \frac{1}{1600}$

Let $P$ = Crippling load by Rankine's formula

Using equation (19.9), we have

$$P = \frac{\sigma_c \cdot A}{1 + \left(\frac{L_e}{k}\right)^2} = \frac{550 \times 225\pi}{1 + \frac{1}{1600} \times \left(\frac{1500}{25.625}\right)^2}$$

$$= \frac{550 \times 225\pi}{3.1415} = \mathbf{123750 \text{ N.}} \quad \textbf{Ans.}$$

**Problem 19.14.** *A hollow cylindrical cast iron column is 4 m long with both ends fixed. Determine the minimum diameter of the column if it has to carry a safe load of 250 kN with a factor of safety of 5. Take the internal diameter as 0.8 times the external diameter. Take $\sigma_c = 550$ N/mm² and $a = \frac{1}{1600}$ in Rankine's formula.* (AMIE, Winter 1983)

**Sol.** Given :

Length of column, $l$ = 4 m = 4000 mm

End conditions = Both ends fixed

∴ Effective length, $L_e = \frac{l}{2} = \frac{4000}{2} = 2000$ mm

Safe load, = 250 kN

Factor of safety, = 5

Let External dia., = $D$

Internal dia. = $0.8 \times D$

Crushing stress, $\sigma_c = 550$ N/mm²

Value of '$a$' = $\frac{1}{1600}$ in Rankine's formula

Now factor of safety = $\frac{\text{Cirippling load}}{\text{Safe load}}$ or $5 = \frac{\text{Cirippling load}}{250}$

∴ Crippling load, $P = 5 \times 250 = 1250$ kN = 1250000 N

Area of column, $A = \frac{\pi}{4}[D^2 - (0.8D)^2]$

$$= \frac{\pi}{4}[D^2 - 0.64D^2] = \frac{\pi}{4} \times 0.36D^2 = \pi \times 0.09D^2$$

Moment of Inertia, $I = \frac{\pi}{64}[D^4 - (0.8D^4] = \frac{\pi}{64}[D^4 - 0.4096D^4]$

$$= \frac{\pi}{64} \times 0.5904 \times D^4 = 0.009225 \times \pi \times D^4$$

But $I = A \times k^2$, where $k$ is radius of gyration

$$\therefore \quad k = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.009225 \times \pi D^4}{\pi \times 0.09 \times D^2}} = 0.32D$$

Now using equation (19.9), $P = \dfrac{\sigma_c \, . \, A}{1 + a\left(\dfrac{L_e}{k}\right)^2}$

or
$$1250000 = \frac{550 \times \pi \times 0.09\, D^2}{1 + \dfrac{1}{1600} \times \left(\dfrac{2000}{0.32D}\right)^2} \qquad (\because \; A = \pi \times 0.09D^2)$$

$$\frac{1250000}{550 \times \pi \times 0.09} = \frac{D^2}{1 + \dfrac{24414}{D^2}} \quad \text{or} \quad 8038 = \frac{D^2 \times D^2}{D^2 + 24414}$$

or $\quad 8038D^2 + 8038 \times 24414 = D^4$ or $D^4 - 8038D^2 - 8038 \times 24414 = 0$

or $\quad D^4 - 8038\, D^2 - 196239700 = 0.$

The above equations is a quadratic equation in $D^2$. The solution is

$$\therefore \quad D^2 = \frac{8038 \pm \sqrt{8038^2 + 4 \times 1 \times 196239700}}{2}$$

$$\left(\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{8038 \pm \sqrt{646094 + 784958800}}{2} = \frac{8038 \pm 29147}{2}$$

$$= \frac{8038 + 29147}{2} \text{ (The other root is not possible)}$$

$$= 18592.5 \text{ mm}^2$$

$$\therefore \quad D = \sqrt{18592.5} = 136.3 \text{ mm}$$

$\therefore$ External diameter $= $ **136.3 mm. Ans.**

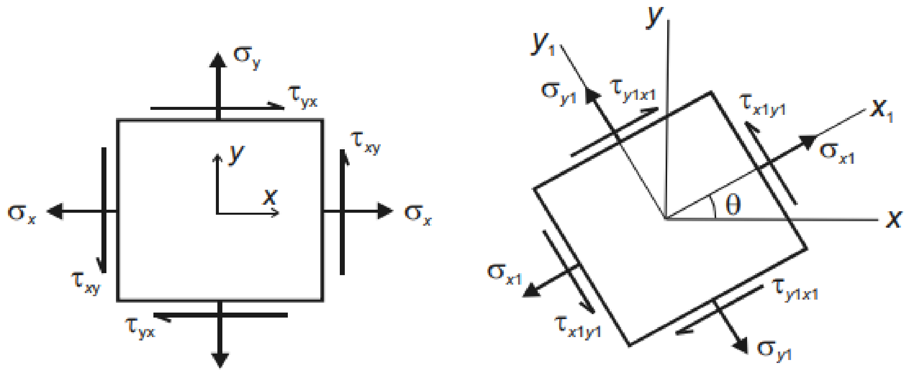
Internal diameter $= 0.8 \times 1363 = $ **109 mm. Ans.**

# MOHR'S CIRCLE IN 2D

## Introduction

- The transformation equations for plane stress can be represented in graphical form by a plot known as Mohr's Circle.
- This graphical representation is extremely useful because it enables you to visualize the relationships between the normal and shear stresses acting on various inclined planes at a point in a stressed body.
- Using Mohr's Circle you can also calculate principal stresses, maximum shear stresses and stresses on inclined planes.

# Stress Transformation Equations




$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

If we vary $\theta$ from 0° to 360°, we will get all possible values of $\sigma_{x1}$ and $\tau_{x1y1}$ for a given stress state. It would be useful to represent $\sigma_{x1}$ and $\tau_{x1y1}$ as functions of $\theta$ in graphical form.

To do this, we must re-write the transformation equations.

$$\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Eliminate θ by squaring both sides of each equation and adding the two equations together.

$$\left(\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x1y1}{}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

Define $\sigma_{avg}$ and $R$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Substitue for $\sigma_{avg}$ and $R$ to get

$$(\sigma_{x1} - \sigma_{avg})^2 + \tau_{x1y1}^2 = R^2$$

which is the equation for a **circle** with centre ($\sigma_{avg}$,0) and radius $R$.

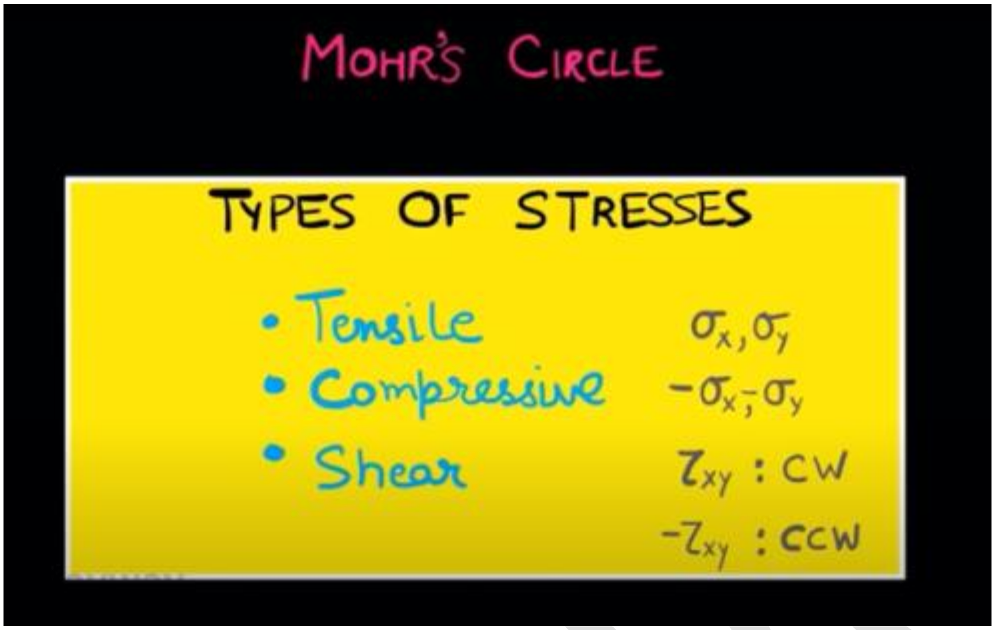
This circle is usually referred to as **Mohr's circle**, after the German civil engineer Otto Mohr (1835-1918). He developed the graphical technique for drawing the circle in 1882.

The construction of Mohr's circle is one of the few graphical techniques still used in engineering. It provides a simple and clear picture of an otherwise complicated analysis.



## HOW TO DRAW - MOHR'S CIRCLE ???

## SIGN CONVENTION




## Construction of Mohr's circle

Assuming we know the stress components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ at a point $P$ in the object under study, as shown in Figure 4, the following are the steps to construct the Mohr circle for the state of stresses at $P$:

1. **Draw the Cartesian coordinate system** $(\sigma_\mathrm{n}, \tau_\mathrm{n})$ with a horizontal $\sigma_\mathrm{n}$-axis and a vertical $\tau_\mathrm{n}$-axis.
2. **Plot two points** $A(\sigma_y, \tau_{xy})$ and $B(\sigma_x, -\tau_{xy})$ in the $(\sigma_\mathrm{n}, \tau_\mathrm{n})$ space corresponding to the known stress components on both perpendicular planes $A$ and $B$, respectively (Figure 4 and 6), following the chosen sign convention.
3. **Draw the diameter of the circle** by joining points $A$ and $B$ with a straight line $\overline{AB}$.
4. **Draw the Mohr Circle**. The centre $O$ of the circle is the midpoint of the diameter line $\overline{AB}$, which corresponds to the intersection of this line with the $\sigma_\mathrm{n}$ axis.

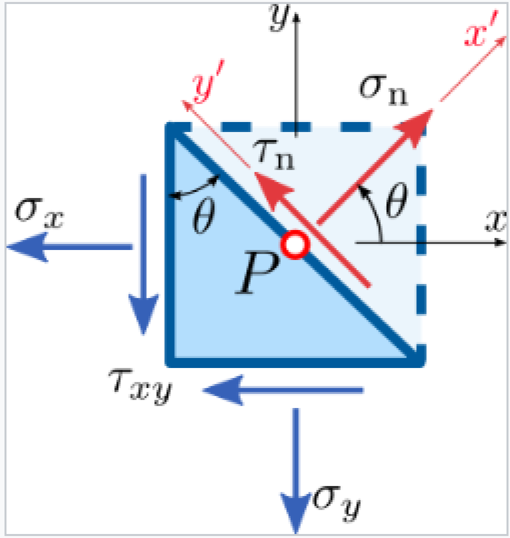
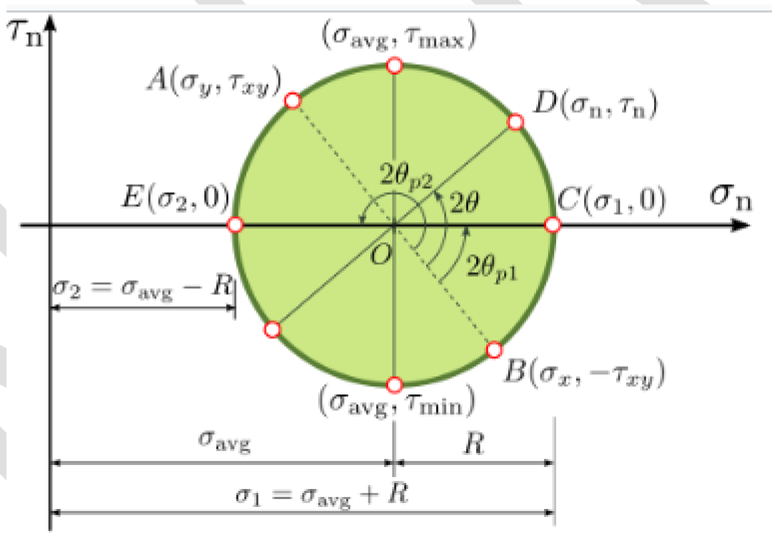



Figure 4. Stress components at a plane passing through a point in a continuum under plane stress conditions.




$$\tau_n = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$R = \sqrt{\left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2}$$

$$\sigma_{avg} = \frac{1}{2}(\sigma_1 + \sigma_2) \qquad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

**Fig:6**

## Finding principal normal stresses from the above figure (Mohr's circle)

The magnitude of the principal stresses are the abscissas of the points $C$ and $E$ (Figure 6) where the circle intersects the $\sigma_\mathrm{n}$-axis. The magnitude of the major principal stress $\sigma_1$ is always the greatest absolute value of the abscissa of any of these two points. Likewise, the magnitude of the minor principal stress $\sigma_2$ is always the lowest absolute value of the abscissa of these two points. As expected, the ordinates of these two points are zero, corresponding to the magnitude of the shear stress components on the principal planes. Alternatively, the values of the principal stresses can be found by

$$\sigma_1 = \sigma_{\max} = \sigma_{\text{avg}} + R$$
$$\sigma_2 = \sigma_{\min} = \sigma_{\text{avg}} - R$$

where the magnitude of the **average normal stress** $\sigma_{\text{avg}}$ is the abscissa of the centre $O$, given by

$$\sigma_{\text{avg}} = \frac{1}{2}(\sigma_x + \sigma_y)$$

and the length of the **radius** $R$ of the circle (based on the equation of a circle passing through two points), is given by

$$R = \sqrt{\left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2}$$

## Finding maximum and minimum shear stresses [edit]

The maximum and minimum shear stresses correspond to the ordinates of the highest and lowest points on the circle, respectively. These points are located at the intersection of the circle with the vertical line passing through the center of the circle, $O$. Thus, the magnitude of the maximum and minimum shear stresses are equal to the value of the circle's radius $R$

$$\tau_{\max,\min} = \pm R$$